

SUPERSYMMETRIC VERTEX ALGEBRAS

REIMUNDO HELUANI AND VICTOR G. KAC

ABSTRACT. We define and study the structure of *SUSY Lie conformal and vertex algebras*. This leads to effective rules for computations with superfields.

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1. INTRODUCTION

1.1. Vertex algebras were introduced about 20 years ago by Borchers [Bor86]. They provide a rigorous definition of the chiral part of 2-dimensional conformal field theory, intensively studied by physicists. Since then, they have had important applications to string theory and conformal field theory, and to mathematics, by providing tools to study the most interesting representations of infinite dimensional Lie algebras. Since their appearance, they have been extensively studied in many papers and books (for the latter we refer to [FLM88], [FHL93], [Kac96], [Hua97], [FBZ01], [BD04]).

1.2. The purpose of the present paper is to define and study the structure of supersymmetric (SUSY) vertex algebras. This theory encompasses the formalism of superfields extensively used by physicists (see eg. [Coh87], [DRS90] and references therein).

Recall [Kac96] that a vertex algebra $(V, |0\rangle, T, Y)$ is a vector superspace V (space of states) with an even vector $|0\rangle$ (vacuum vector), an even endomorphism T (translation operator), and a parity preserving bilinear product with values in Laurent series in an indeterminate z over V :

$$V \otimes V \rightarrow V((z)), \quad a \otimes b \mapsto Y(a, z)b = \sum_{n \in \mathbb{Z}} (a_{(n)}b)z^{-1-n},$$

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subject to the following axioms ($a, b \in V$):

- (vacuum axioms) $Y(a, z)|0\rangle|_{z=0} = a$, $T|0\rangle = 0$,
- (translation invariance) $[T, Y(a, z)] = \partial_z Y(a, z)$,
- (locality) $(z - w)^N [Y(a, z), Y(b, w)] = 0$ for some $N \in \mathbb{Z}_+$.

The vertex algebras which arise in conformal field theory carry a *conformal vector* $\nu \in V$, so that the coefficients of the corresponding field

$$(1.2.1) \quad Y(\nu, z) \equiv L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-2-n}$$

satisfy the Virasoro commutation relations

$$(1.2.2) \quad [L_m, L_n] = (m - n)L_{m+n} + \delta_{m, -n} \frac{m^3 - m}{12} c,$$

where $c \in \mathbb{C}$ is the *central charge*, and also the following two properties hold:

•

$$(1.2.3) \quad L_{-1} = T,$$

- L_0 is diagonalizable on V with eigenvalues bounded below.

Sometimes, such a vertex algebra V (called *conformal*) admits a “supersymmetry”, namely, V carries a *superconformal vector* τ , such that the corresponding field

$$(1.2.4) \quad Y(\tau, z) \equiv G(z) = \sum_{n \in \frac{1}{2} + \mathbb{Z}} G_n z^{-3/2-n}$$

satisfies the following properties:

- (1) $\nu = \frac{1}{2}G_{-1/2}\tau$ is a conformal vector with central charge c ;
- (2) the operators G_n ($n \in 1/2 + \mathbb{Z}$) along with the Virasoro operators L_n ($n \in \mathbb{Z}$), appearing in $L(z) = Y(\nu, z)$, form the Neveu-Schwarz algebra with central charge c , namely, along with the Virasoro relations (1.2.2), the following commutation relations hold:

$$(1.2.5) \quad [G_m, L_n] = \left(m - \frac{n}{2}\right) G_{m+n}, \quad [G_m, G_n] = 2L_{m+n} + \frac{c}{3} \left(m^2 - \frac{1}{4}\right) \delta_{m, -n}.$$

Then V is called an $N = 1$ superconformal vertex algebra. In particular, we then can obtain an enhanced translation invariance property as follows. Let $S = G_{-1/2}$, let θ be an odd indeterminate, commuting with z : $\theta^2 = 0$, $\theta z = z\theta$, and to each $a \in V$ associate a *superfield*:

$$(1.2.6) \quad Y(a, z, \theta) = Y(a, z) + \theta Y(Sa, z).$$

Then one can show, using the so called commutator formula, that

$$(1.2.7) \quad [S, Y(a, z, \theta)] = (\partial_\theta - \theta \partial_z)(a, z, \theta).$$

Since, by the second formula in (1.2.5) and (1.2.3), we have

$$(1.2.8) \quad S^2 = T,$$

formula (1.2.7) implies the usual translation invariance $[T, Y(a, z, \theta)] = \partial_z Y(a, z, \theta)$.

This leads to the following definition of an $N_K = 1$ *SUSY vertex algebra* $(V, |0\rangle, S, Y)$, where S is an odd endomorphism of the space of states V , and Y is a parity preserving bilinear product with values in $V((z))[\theta]$:

$$V \otimes V \rightarrow V((z))[\theta], \quad a \otimes b \mapsto Y(a, z, \theta)b = \sum_{\substack{n \in \mathbb{Z} \\ i=0,1}} \theta^{1-i} z^{-1-n} a_{(n|i)} b,$$

subject to the following axioms:

- (vacuum axioms) $Y(a, z, \theta)|0\rangle|_{z=0, \theta=0} = a, S|0\rangle = 0,$
- (translation invariance) $[S, Y(a, z, \theta)] = (\partial_\theta - \theta \partial_z)Y(a, z, \theta),$
- (locality) $(z - w)^N [Y(a, z, \theta), Y(b, w, \zeta)] = 0$ for some $N \in \mathbb{Z}_+.$

This SUSY vertex algebra is called *conformal* if there exists an odd vector $\tau \in V$, such that

$$Y(\tau, z, \theta) = G(z) + 2\theta L(z),$$

and such that the coefficients of the expansions of these fields as in (1.2.1) and (1.2.4), satisfy the commutation relations (1.2.2) and (1.2.5) of the Neveu-Schwarz algebra, $S = G_{-1/2}$ and L_0 is diagonalizable with eigenvalues bounded below.

Of course, a SUSY vertex algebra $(V, |0\rangle, S, Y(a, z, \theta))$ gives rise to an ordinary vertex algebra $(V, |0\rangle, T = S^2, Y(a, z) = Y(a, z, 0))$, and a superconformal vector τ gives rise to a conformal vector $\nu = \frac{1}{2}S\tau$. However, computations with superfields are much more effective than with the ordinary fields.

1.3. Now, the Neveu-Schwarz is the simplest, after Virasoro, among the superconformal Lie algebras. A *superconformal Lie algebra* is a pair $(\mathcal{L}, \mathcal{F})$, where \mathcal{L} is a Lie superalgebra and

$$\mathcal{F} = \left\{ a^i(z) = \sum_{n \in \mathbb{Z}} a_n^i z^{-1-n} \right\}_{i \in I}$$

is a finite family of formal distributions whose coefficients a_n^i span \mathcal{L} , satisfying the following properties:

- (1) \mathcal{F} contains a Virasoro formal distribution $L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-2-n}$, such that $[L_{-1}, a^i(z)] = \partial_z a^i(z);$
- (2) the formal distributions $a^i(z)$ are pairwise local: $(z - w)^N [a^i(z), a^j(w)] = 0$ for some $N \in \mathbb{Z}_+$, or, equivalently:

$$[a^i(z), a^j(w)] = \sum_{s=0}^{N-1} c_s^{ij}(w) \partial_w^s \delta(z, w)$$

for some formal distributions $c_s^{ij}(z);$

- (3) $c_s^{ij}(z) \in \mathbb{C}[\partial_z] \mathcal{F};$
- (4) \mathcal{L} is a non-split central extension of an almost simple Lie superalgebra $\tilde{\mathcal{L}}$ (i.e. all non-zero ideals of $\tilde{\mathcal{L}}$ contain its derived subalgebra $\tilde{\mathcal{L}}'$).

1.4. A complete list of centerless superconformal Lie algebras consists of four series ($N \in \mathbb{Z}_+$): $W(1|N)$, $S(1|N+2; a)$, $\tilde{S}(1|N+2)$, $K(1|N)$, and two exceptions $K(1|4)'$ and $CK(1|6)$ (see [FK02]). Here $W(1|N)$ denotes the Lie superalgebra of all derivations of $\mathbb{C}[z, z^{-1}, \theta^1, \dots, \theta^N]$, where z is an even indeterminate and θ^i 's are odd anticommuting indeterminates, commuting with z (in particular, $W(1|0)$ is the centerless Virasoro algebra). The Lie superalgebras $S(1|N; a)$ and $\tilde{S}(1|N)$ (resp.

$K(1|N)$) are subalgebras of $W(1|N)$ consisting of vector fields, annihilating certain supervolume forms (resp. preserving the super-contact form¹ $dz + \sum_{i=1}^N \theta^i d\theta^i$, up to multiplication by a function), $K(1|4)'$ is the derived algebra of $K(1|4)$, and $CK(1|6)$ is a certain subalgebra of $K(1|6)$. The only algebras on the list that admit a central extension are $W(1|N)$ with $N \leq 1$, $S(1|2; a)$, $\tilde{S}(1|2)$, $K(1|N)$ with $N \leq 4$ and $K(1|4)'$ (see [KvdL89] and [FK02]). Note that the Neveu-Schwarz algebra is a central extension of $K(1|1)$.

The subalgebra $\mathcal{L}_{\leq} = \text{span}\{a_n^i | i \in I, n \geq 0\}$ is called the annihilation subalgebra of \mathcal{L} . The reason for this name comes from the fact that the space

$$(1.4.1) \quad V(\mathcal{L}) = U(\mathcal{L})/U(\mathcal{L})\mathcal{L}_{\leq},$$

where $U(\mathcal{L})$ denotes the universal enveloping superalgebra of \mathcal{L} , carries a canonical structure of a vertex algebra, called the universal enveloping vertex algebra of \mathcal{L} , where $|0\rangle$ is the image of 1, T is induced by L_{-1} , and the image of \mathcal{F} is contained in the space of fields of $V(\mathcal{L})$, so that $\mathcal{L}_{\leq}|0\rangle = 0$.

In all cases the subalgebra \mathcal{L}_{\leq} consists of all vector fields of \mathcal{L} which are regular at the origin. Denote by $\mathcal{L}_{<}$ the subalgebra of \mathcal{L}_{\leq} , consisting of vector fields, which vanish at the origin. Then in all cases we can find a complementary subalgebra \mathcal{L}_{tr} (the “translation” subalgebra)

$$(1.4.2) \quad \mathcal{L}_{\leq} = \mathcal{L}_{\text{tr}} \oplus \mathcal{L}_{<}.$$

In the cases W and S one can choose $\mathcal{L}_{\text{tr}} = \text{span}_{\mathbb{C}}\{\partial_z, \partial_{\theta^1}, \dots, \partial_{\theta^N}\}$. However, in the remaining cases the simplest choice is

$$\mathcal{L}_{\text{tr}} = \text{span}_{\mathbb{C}}\{\partial_z, \partial_{\theta^1} - \theta^1 \partial_z, \dots, \partial_{\theta^N} - \theta^N \partial_z\}.$$

Denote these Lie superalgebras by W_{tr} and K_{tr} respectively, and by \mathcal{H}_W (resp. \mathcal{H}_K) the universal enveloping superalgebra of W_{tr} (resp. K_{tr}). It is an associative algebra on one even generator T and N odd generators S^i , subject to the relations:

$$[T, S^i] = 0, \quad [S^i, S^j] = 0 \text{ (resp. } = 2\delta_{i,j}T\text{)}.$$

Note that the operators T and S of an $N_K = 1$ SUSY vertex algebra define a representation of the associative superalgebra \mathcal{H}_K for $N = 1$ (see 1.2.8).

This leads to the definition of an N_W (resp. $N_K = N$) SUSY vertex algebra as a \mathcal{H}_W (resp. \mathcal{H}_K)-module V with the vacuum vector $|0\rangle$ and a parity preserving bilinear product with values in $V((z))[\theta^1, \dots, \theta^N]$,

$$a \otimes b \mapsto Y(a, z, \theta^1, \dots, \theta^N) = \sum_{n \in \mathbb{Z}, J} \theta^{J^c} z^{-1-n} a_{(n|J)} b,$$

where J runs over all ordered subsets of $\{1, \dots, N\}$, J^c denotes the ordered complement of J in $\{1, \dots, N\}$ and $\theta^J = \theta^{j_1} \dots \theta^{j_k}$ for $J = \{j_1, \dots, j_k\}$, subject to the vacuum, translation invariance and locality axioms. The vacuum and locality axioms generalize in the obvious way; translation invariance means the following:

$$[T, Y(a, z, \theta)] = \partial_z Y(a, z, \theta),$$

$$[S^i, Y(a, z, \theta)] = \partial_{\theta^i} Y(a, z, \theta) \quad \text{in the } N_W = N \text{ case,}$$

$$[S^i, Y(a, z, \theta)] = (\partial_{\theta^i} - \theta^i \partial_z) Y(a, z, \theta) \quad \text{in the } N_K = N \text{ case.}$$

¹Here and further we use the sign convention of [DM99], which differs from the one used in [KvdL89] and [FK02]. The main difference is that the de Rham differential d is an *even* derivation.

1.5. Given a superconformal Lie algebra \mathcal{L} , and a decomposition (1.4.2) of its annihilation subalgebra, such that $\mathcal{L}_{\text{tr}} \simeq W_{\text{tr}}$ (resp. $\simeq K_{\text{tr}}$), we define an \mathcal{L} -conformal N_W (resp. N_K) = N SUSY vertex algebra V by the property that V carries a representation of \mathcal{L} such that the representation of $U(\mathcal{L}_{\text{tr}})$ coincides with that of \mathcal{H}_W (resp. \mathcal{H}_K), the formal distributions of \mathcal{L} are represented by fields of V , and L_0 is diagonalizable with eigenvalues bounded below. The “minimal” example of an \mathcal{L} -superconformal vertex algebra is $V(\mathcal{L})$, given by (1.4.1), or $V^c(\mathcal{L}) = V(\mathcal{L})/(C - c1)V(\mathcal{L})$ in the case \mathcal{L} has a central element C , called the *SUSY Virasoro* vertex algebra associated to \mathcal{L} .

1.6. We develop the structure theory of SUSY vertex algebras along the lines of [Kac96].

First, we develop the calculus of formal superdistributions. As usual, the key role is played by the formal super delta-function

$$(1.6.1) \quad \delta(Z, W) = (\theta^1 - \zeta^1) \dots (\theta^N - \zeta^N) (i_{z,w} - i_{w,z}) \frac{1}{z - w},$$

where $Z = (z, \theta^1, \dots, \theta^N)$, $W = (w, \zeta^1, \dots, \zeta^N)$, and $i_{z,w}$ signifies the expansion in the domain $|z| > |w|$. The key result is the decomposition formula of a local formal superdistribution:

$$(1.6.2) \quad a(Z, W) = \sum_{j \geq 0, J} \partial_W^{(j|J)} \delta(Z, W) c_{j|J}(W),$$

where

$$(1.6.3) \quad c_{j|J}(W) = \text{res}_Z (Z - W)^{j|J} a(Z, W),$$

and res_Z stands for the coefficient of $\theta^1 \dots \theta^N z^{-1}$. There are actually two cases, W and K , of this formula, depending on the choice of $\partial_W^{j|J}$ in (1.6.2) and the respective choice of $(Z - W)^{j|J}$ in (1.6.3).

We let for $J = \{j_1 < \dots < j_s\}$:

$$\begin{aligned} \partial_W^{j|J} &= \partial_w^j \partial_{\theta^{j_1}} \dots \partial_{\theta^{j_s}} && \text{in case } W, \\ \partial_W^{j|J} &= \partial_w^j (\partial_{\theta^{j_1}} + \theta^{j_1} \partial_w) \dots (\partial_{\theta^{j_s}} + \theta^{j_s} \partial_w) && \text{in case } K, \end{aligned}$$

and in both cases we let $\partial_W^{(j|J)} = (-1)^{|J|(|J|+1)/2} \partial^{j|J} / j!$. (The second choice will be denoted by $D_W^{j|J}$.) Respectively, we let

$$\begin{aligned} (Z - W)^{j|J} &= (z - w)^j (\theta^{j_1} - \zeta^{j_1}) \dots (\theta^{j_s} - \zeta^{j_s}) && \text{in case } W, \\ (Z - W)^{j|J} &= \left(z - w - \sum_{i=1}^N \theta^i \zeta^i \right)^j (\theta^{j_1} - \zeta^{j_1}) \dots (\theta^{j_s} - \zeta^{j_s}) && \text{in case } K. \end{aligned}$$

1.7. Next, the formal Fourier transform is defined by

$$\mathcal{F}_{Z,W}^\Lambda a(Z, W) = \text{res}_Z \exp((Z - W)\Lambda) a(Z, W),$$

where $\Lambda = (\lambda, \chi^1, \dots, \chi^N)$, λ is an even indeterminate, χ^i 's are odd indeterminates, commuting with λ and satisfying the following commutation relations:

$$\begin{aligned} \chi^i \chi^j + \chi^j \chi^i &= 0 && \text{in case } W, \\ \chi^i \chi^j + \chi^j \chi^i &= -2\delta_{i,j} \lambda && \text{in case } K. \end{aligned}$$

Defining the Λ -*bracket*

$$[a(W)_\Lambda b(W)] = \mathcal{F}_{Z,W}^\Lambda[a(Z), b(W)]$$

of formal superdistributions, we arrive at the notion of the N_W (resp. N_K) = N SUSY Lie conformal algebra \mathcal{R} , which, as usual, encodes the singular part of the operator product expansion (OPE) of a local pair of superfields. The case $N = 0$ is that of an ordinary Lie conformal superalgebra [Kac96].

The new phenomenon for $N > 0$ is that the Λ -bracket has parity $N \bmod 2$. Consequently, the bracket $[a_\Lambda b]|_{\Lambda=0}$ induces on $\mathcal{R}/(T\mathcal{R} + \sum_i S^i \mathcal{R})$ a structure of a Lie superalgebra of parity $N \bmod 2$.

On the other hand, the structure of a SUSY Lie conformal algebra is an important part of the structure of a SUSY vertex algebra. As in the case of vertex algebras, the only missing ingredient is the normally ordered product of superfields, which is defined as usual:

$$: a(Z)b(Z) := a_+(Z)b(Z) + (-1)^{p(a)p(b)}b(Z)a_-(Z),$$

where for a superfield $a(Z) = \sum_{n \in \mathbb{Z}, J} \theta^{J^c} z^{-1-n} a_{(n|J)}$ we let

$$a_-(Z) = \sum_{n \geq 0} \theta^{J^c} z^{-1-n} a_{(n|J)}, \quad a_+(Z) = a(Z) - a_-(Z).$$

We prove that the non-commutative Wick formula, which allows to compute the singular part of the OPE for normally ordered products, generalizes to the SUSY $N_W = N$ and $N_K = N$ cases almost verbatim:

$$(1.7.1) \quad [a_\Lambda : bc :] = : [a_\Lambda b] c : + (-1)^{p(a)+N} b : [a_\Lambda c] : + \int_0^\Lambda [[a_\Lambda b]_{\Gamma c}] d\Gamma.$$

Furthermore, we show that the uniqueness and existence theorems, as well as all basic identities for vertex algebras, extend to the SUSY case with minor modification of signs. In particular, we show that a N_W (resp. N_K) = N SUSY vertex algebra is a N_W (resp. N_K) = N SUSY Lie conformal algebra with the Λ bracket:

$$[a_\Lambda b] = \text{res}_Z e^{\Lambda Z} Y(a, Z)b,$$

together with a unital, “quasicommutative” and “quasiassociative” differential superalgebra structure $::$, which are related by formula (1.7.1). This is an equivalent definition of a SUSY vertex algebra, analogous to the one given in [BK03] for vertex algebras. Removing “quantum corrections”, we obtain the definition of a SUSY Poisson vertex algebra.

1.8. As in the vertex algebra case, most of the basic examples of SUSY vertex algebras are constructed starting with a SUSY Lie conformal algebra \mathcal{R} , and taking its universal enveloping vertex algebra $V(\mathcal{R})$ or $V^c(\mathcal{R})$. This can be defined as the universal enveloping vertex algebra $V(\mathcal{L})$ of the corresponding formal superdistribution Lie algebra $V(\mathcal{L})$ (resp. $V^c(\mathcal{L})$), defined in the same way as in (1.4.1). By abuse of terminology, we often say that $V(\mathcal{R})$ is a SUSY vertex algebra generated by \mathcal{R} .

The simplest example of an $N_K = N$ SUSY vertex algebra is the well-known boson-fermion system, generated by one superfield Ψ of parity $N \bmod 2$, subject to the following Λ -bracket:

$$[\Psi_\Lambda \Psi] = \Lambda^{1|N} \text{ for even } N, \quad [\Psi_\Lambda \Psi] = \Lambda^{0|N} \text{ for odd } N.$$

Viewed as an ordinary vertex algebra, this SUSY vertex algebra in the case $N = 1$ is generated by one boson and one fermion (hence the name “boson-fermion” system).

Another example is the SUSY vertex algebra, generated by the current SUSY Lie conformal algebra, associated to a Lie superalgebra \mathfrak{g} with an invariant bilinear form (\cdot, \cdot) . For N even, the corresponding SUSY $N_W = N$ or $N_K = N$ Λ -bracket is defined as

$$[a_\Lambda b] = [a, b] + \lambda(a, b), \quad a, b \in \mathfrak{g}.$$

For N odd, we consider the vector superspace $\Pi\mathfrak{g}$ with reversed parity, and define for $\bar{a}, \bar{b} \in \Pi\mathfrak{g}$:

$$[\bar{a}_\Lambda \bar{b}] = (-1)^{p(a)} \left(\overline{[a, b]} + (a, b) \sum_{i=1}^N \chi^i \right).$$

In the case $N_K = 1$, using the normally ordered products of the above superfields, one reproduces the construction of the $N_K = 1$ super Virasoro field from [KT85].

Unfortunately, we do not know how to construct a SUSY lattice vertex algebra.

A less routine example is the following. Let \mathfrak{g} be a Lie algebra and F a \mathfrak{g} -module. We associate to this data an $N_K = 1$ SUSY Lie conformal algebra $\mathcal{R}(\mathfrak{g}, F)$, which is a free \mathcal{H}_K -module over the vector superspace:

$$(F \oplus \mathfrak{g}^*) \oplus (\mathfrak{g} \oplus \Pi\mathfrak{g}^*),$$

where $(F \oplus \mathfrak{g}^*)$ and $(\mathfrak{g} \oplus \Pi\mathfrak{g}^*)$ are the even and odd parts respectively and Π is the change of parity functor, with the following non-zero Λ -brackets ($X, Y \in \mathfrak{g}$, $f \in F$, $\alpha \in \mathfrak{g}^*$, $\bar{\alpha} \in \Pi\mathfrak{g}^*$):

$$\begin{aligned} [X_\Lambda Y] &= [X, Y], & [X_\Lambda f] &= Xf, \\ [X_\Lambda \alpha] &= X\alpha + \lambda \langle \alpha, X \rangle, & [X_\Lambda \bar{\alpha}] &= \overline{X\alpha} + \chi \langle \alpha, X \rangle. \end{aligned}$$

The corresponding SUSY universal enveloping vertex algebra is denoted by $V(\mathfrak{g}, F)$. The SUSY $N_K = 1$ vertex algebra $V(\mathfrak{g}, F)$ in the case when \mathfrak{g} is the Lie algebra of vector fields on a manifold M , F is the space of functions on M and \mathfrak{g}^* is the space of differential 1-forms on M , is used in [BZHS06] to construct the chiral de Rham complex [MSV99], as a sheaf of $N_K = 1$ SUSY vertex algebras on M , and study its SUSY properties.

1.9. In the subsequent paper [Hel06] the formalism of SUSY vertex algebras is applied to associate to any (strongly)conformal SUSY N_W (resp N_K) = N vertex algebra V a vector bundle \mathcal{V}_X on any supercurve (resp. superconformal curve) X , along the lines of [FBZ01], where this is done for ordinary curves.

2. PRELIMINARIES

In this section we recall some notation and basic results on vertex algebras. We also give the first examples of SUSY vertex algebras constructed via ordinary vertex algebras. The reader is referred to [Kac96] for an introduction to the vertex algebra theory.

Definition 2.1. Let \mathcal{A} be a Lie superalgebra. An \mathcal{A} -valued formal distribution is a formal expression of the form:

$$B(z) = \sum_{n \in \mathbb{Z}} B_{(n)} z^{-1-n}$$

where $B_{(n)} \in \mathcal{A}$ have the same parity for all $n \in \mathbb{Z}$; this parity is called the *parity of $B(z)$* . The coefficients $B_{(n)}$ are called the *Fourier modes* of $B(z)$, and z is an indeterminate. A pair of formal distributions $B(z)$, $C(w)$ is *local* if

$$(z - w)^N [B(z), C(w)] = 0 \quad \text{for some } N \in \mathbb{Z}_+.$$

If $\mathcal{A} = \text{End}(V)$, where V is a vector superspace, with the usual superbracket, we say that $B(z)$ is a *field* if, for every $v \in V$, $B_{(n)}v = 0$ for large enough n .

2.2. Let V be a vertex algebra² (see 1.2). The map Y is called the *state-field correspondence* and we will use this map to identify a vector $a \in V$ with its corresponding field $Y(a, z)$.

2.3. Given a vertex algebra V we denote

$$(2.3.1) \quad a_{(n)}b = a_{(n)}(b), \quad [a_\lambda b] = \sum_{k \geq 0} \frac{\lambda^k}{k!} a_{(k)}b, \quad :ab := a_{(-1)}b.$$

The first operation is called the n -th product, the second is called the λ -bracket and the third the normally ordered product.

2.4. For each $n \in \mathbb{Z}$, define the n -th product of $\text{End}(V)$ -valued fields $A(z)$ and $B(z)$ as follows. Denote by $i_{z,w}$ the *expansion in the domain $|z| > |w|$* :

$$(2.4.1) \quad i_{z,w} z^m w^n (z - w)^k = z^{m+k} w^n i_{z,w} \left(1 - \frac{w}{z}\right)^k = \sum_{j \geq 0} (-1)^j \binom{k}{j} z^{m+k-j} w^{n+j}.$$

Define

$$(2.4.2) \quad A(w)_{(n)}B(w) = \text{res}_z (i_{z,w}(z - w)^n A(z)B(w) - i_{w,z}(z - w)^n (-1)^{p(A)p(B)} B(w)A(z)),$$

where $p(A)$ denotes the parity of $A(w)$. It can be shown that the following n -th product identity holds (cf. [Kac96, Prop. 4.4])

$$(2.4.3) \quad Y(a_{(n)}b, z) = Y(a, z)_{(n)}Y(b, z) \quad \forall n \in \mathbb{Z},$$

hence,

$$(2.4.4) \quad Y(Ta, z) = \partial_z Y(a, z).$$

2.5. In a vertex algebra V we have the following commutator formulas [Kac96, p 112]

$$\begin{aligned} [a_{(m)}, b_{(n)}] &= \sum_{j \geq 0} \binom{m}{j} (a_{(j)}b)_{(m+n-j)}, \\ [a_{(m)}, Y(b, w)] &= \sum_{j \geq 0} \left(\frac{\partial_w^j w^m}{j!} \right) Y(a_{(j)}b, w). \end{aligned}$$

This formula shows that the space of Fourier modes of all fields of a vertex algebra is closed under the Lie bracket, and, moreover, the commutation relations are expressed in terms of j -th products.

²We will denote a vertex algebra by its space of states V when there is no possible confusion.

Definition 2.6. A *Lie conformal algebra* is a super $\mathbb{C}[\partial]$ -module \mathcal{R} equipped with a parity preserving bilinear map

$$[\lambda] : \mathcal{R} \otimes \mathcal{R} \rightarrow \mathbb{C}[\lambda] \otimes \mathcal{R}, \quad a \otimes b \mapsto [a_\lambda b],$$

satisfying the following axioms:

- Sesquilinearity:

$$[\partial a_\lambda b] = -\lambda [a_\lambda b], \quad [a_\lambda \partial b] = (\partial + \lambda) [a_\lambda b].$$

- Skew-commutativity:

$$[b_\lambda a] = -(-1)^{p(a)p(b)} [a_{-\partial-\lambda} b].$$

- Jacobi identity:

$$[a_\lambda [b_\mu c]] = [[a_\lambda b]_{\lambda+\mu} c] + (-1)^{p(a)p(b)} [b_\mu [a_\lambda c]],$$

for all $a, b, c \in \mathcal{R}$. Here λ and μ are commuting indeterminates.

Given a Lie conformal algebra \mathcal{R} , we can associate to it a vertex algebra $V(\mathcal{R})$ (cf. [Kac96], [BK03]) called the *universal enveloping vertex algebra of \mathcal{R}* , as defined in the introduction. If \mathcal{R} is generated by some vectors $\{a_i\}$ as a $\mathbb{C}[\partial]$ -module, we say that $V(\mathcal{R})$ is generated by the same vectors. If $C \in \mathcal{R}$ is a central element such that $\partial C = 0$, given any complex number c , we denote by $V^c(\mathcal{R})$ the quotient of $V(\mathcal{R})$ by the ideal $(C - c)V(\mathcal{R})$.

One can show [Kac96] that a vertex algebra V is canonically a Lie conformal algebra with the λ -bracket defined in (2.3.1) and $\partial = T$.

Example 2.7. The *Virasoro* vertex algebra Vir^c is generated by an even field L satisfying:

$$(2.7.1) \quad [L_\lambda L] = (\partial + 2\lambda)L + \frac{c}{12}\lambda^3.$$

The complex number c is called the *central charge*. Expanding the corresponding field as in (1.2.1) we obtain the familiar commutation relations of the Virasoro algebra (1.2.2).

2.8. Let $\nu \in V$ be a conformal vector (see 1.2) and let $L(z)$ be the corresponding Virasoro field. A vector $a \in V$ satisfying $[L_\lambda a] = (T + \Delta\lambda)a + O(\lambda^2)$ is said to have *conformal weight Δ* . If, moreover, a satisfies $[L_\lambda a] = (T + \Delta\lambda)a$ we say that a is *primary*.

Example 2.9. The Neveu Schwarz (NS) vertex algebra is generated by an even Virasoro field L (satisfying (2.7.1)) and an odd primary field G of conformal weight $3/2$, satisfying the commutation relation:

$$[G_\lambda G] = 2L + \frac{\lambda^2}{3}c.$$

If we expand the corresponding fields as in (1.2.1) and (1.2.4) we obtain the commutation relations (1.2.5) and (1.2.2) of the Neveu-Schwarz algebra.

As we have seen in the introduction, given a vertex algebra with an $N = 1$ superconformal vector τ , we obtain an $N_K = 1$ SUSY vertex algebra (see also 4.17 for a definition) by defining the superfields (1.2.6). In particular, the Neveu-Schwarz algebra gives rise to such an $N_K = 1$ SUSY vertex algebra.

Below we give some examples of vertex algebras with an $N = 1$ superconformal vector. By the construction in 1.2, they are automatically $N_K = 1$ SUSY vertex algebras.

Example 2.10. [Kac96, Ex. 5.9a] Let V be the universal enveloping vertex algebra of the Lie conformal algebra generated by an even vector (free boson) α and an odd vector (free fermion) φ , namely

$$[\alpha_\lambda \alpha] = \lambda, \quad [\varphi_\lambda \varphi] = 1, \quad [\alpha_\lambda \varphi] = 0.$$

Then V is a (simple) vertex algebra with a family of $N = 1$ superconformal vectors

$$\tau = (\alpha_{(-1)}\varphi_{(-1)} + m\varphi_{(-2)})|0\rangle, \quad m \in \mathbb{C},$$

of central charge $c = \frac{3}{2} - 3m^2$.

Example 2.11. [KT85] [Kac96, Thm 5.9] Let \mathfrak{g} be a finite dimensional Lie algebra with a non-degenerate invariant symmetric bilinear form (\cdot, \cdot) , normalized by the condition $(\alpha, \alpha) = 2$ for a long root α , and let h^\vee be the dual Coxeter number. We construct a vertex algebra $V^k(\mathfrak{g}_{\text{super}})$ generated by the usual currents $a, b \in \mathfrak{g}$, satisfying:

$$[a_\lambda b] = [a, b] + (k + h^\vee)\lambda(a, b),$$

and the odd super currents $\bar{a} \in \Pi\mathfrak{g}$ (as before Π stands for reversal of parity), satisfying:

$$[a_\lambda \bar{b}] = \overline{[a, b]}, \quad [\bar{a}_\lambda \bar{b}] = (k + h^\vee)(a, b).$$

Let $\{a^i\}$ and $\{b^i\}$ be dual bases of \mathfrak{g} . Provided that $k \neq -h^\vee$ the vertex algebra $V^k(\mathfrak{g}_{\text{super}})$ admits an $N = 1$ superconformal vector

$$\tau = \frac{1}{k + h^\vee} \left(\sum_i a_{(-1)}^i \bar{b}_{(-1)}^i + \frac{1}{3(k + h^\vee)} \sum_{i,j,r} ([a^i, a^j], a^r) \bar{b}_{(-1)}^i \bar{b}_{(-1)}^j \bar{b}_{(-1)}^r \right) |0\rangle,$$

of central charge

$$c_k = \frac{k \dim \mathfrak{g}}{k + h^\vee} + \frac{1}{2} \dim \mathfrak{g}.$$

This is known as the *Kac-Todorov* construction. The formulas in [Kac96] should be corrected as above.

Example 2.12. [Kac96, Thm 5.10] The $N = 2$ vertex algebra is generated by a Virasoro field L of central charge c , an even field J , primary of conformal weight 1, and two odd fields G^\pm , primary of conformal weight $3/2$. The remaining commutation relations are:

$$[J_\lambda J] = \frac{c}{3}\lambda, \quad [G_\lambda^\pm G^\pm] = 0, \quad [J_\lambda G^\pm] = \pm G^\pm,$$

$$[G_\lambda^+ G^-] = L + \frac{1}{2}\partial J + \lambda J + \frac{c}{6}\lambda^2.$$

This vertex algebra contains an $N = 1$ superconformal vector:

$$\tau = G_{(-1)}^+ |0\rangle + G_{(-1)}^- |0\rangle.$$

Also, this vertex algebra admits a $\mathbb{Z}/2\mathbb{Z} \times \mathbb{C}^*$ family of automorphisms. The generator of $\mathbb{Z}/2\mathbb{Z}$ is given by $L \mapsto L$, $J \mapsto -J$ and $G^\pm \mapsto G^\mp$. The \mathbb{C}^* family is given

by $G^+ \mapsto \mu G^+$ and $G^- \mapsto \mu^{-1} G^-$. Applying these automorphisms, we get a family of $N = 1$ superconformal vectors. By expanding the corresponding fields

$$L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-2-n}, \quad G^\pm(z) = \sum_{n \in 1/2 + \mathbb{Z}} G_n^\pm z^{-3/2-n}, \quad J(z) = \sum_{n \in \mathbb{Z}} J_n z^{-1-n},$$

we get the commutation relations of the Virasoro operators L_n , and the following remaining commutation relations

$$\begin{aligned} [J_m, J_n] &= \frac{m}{3} \delta_{m,-n} c, & [J_m, G_n^\pm] &= \pm G_{m+n}^\pm, & [G_m^\pm, L_n] &= \left(m - \frac{n}{2}\right) G_{m+n}^\pm, \\ [L_m, J_n] &= -n J_{m+n}, & [G_m^+, G_n^-] &= L_{m+n} + \frac{m-n}{2} J_{m+n} + \frac{c}{6} \left(m^2 - \frac{1}{4}\right) \delta_{m,-n}. \end{aligned}$$

Sometimes it is convenient to introduce a different set of generating fields for this vertex algebra. We define $\tilde{L} = L - 1/2 \partial J$. This is a Virasoro field with central charge zero, namely $[\tilde{L}_\lambda \tilde{L}] = (\partial + 2\lambda) \tilde{L}$. With respect to this Virasoro element, G^+ is primary of conformal weight 2 and G^- is primary of conformal weight 1; J has conformal weight 1 but is no longer a primary field. To summarize the commutation relations, we write

$$\begin{aligned} (2.12.1a) \quad Q(z) = G^+(z) &= \sum_{n \in \mathbb{Z}} Q_n z^{-2-n}, & H(z) = G^-(z) &= \sum_{n \in \mathbb{Z}} H_n z^{-1-n}, \\ \tilde{L}(z) &= \sum_{n \in \mathbb{Z}} T_n z^{-2-n}. \end{aligned}$$

The corresponding λ -brackets of these fields are given by:

$$\begin{aligned} (2.12.1b) \quad [\tilde{L}_\lambda \tilde{L}] &= (\partial + 2\lambda) \tilde{L}, & [\tilde{L}_\lambda J] &= (\partial + \lambda) J - \frac{\lambda^2}{6} c, & [\tilde{L}_\lambda Q] &= (\partial + 2\lambda) Q, \\ [\tilde{L}_\lambda H] &= (\partial + \lambda) H, & [H_\lambda Q] &= \tilde{L} - \lambda J + \frac{c}{6} \lambda^2. \end{aligned}$$

The commutation relations between the Fourier coefficients are:

$$\begin{aligned} (2.12.1c) \quad [T_m, T_n] &= (m-n) T_{m+n}, & [Q_m, Q_n] &= [H_m, H_n] = 0, \\ [T_m, H_n] &= -n H_{m+n}, & [T_m, J_n] &= -n J_{m+n} - m(m+1) \frac{c}{12} \delta_{m,-n}, \\ [T_m, Q_n] &= (m-n) Q_{m+n}, & [H_m, Q_n] &= T_{m+n} - m J_{m+n} + m(m-1) \frac{c}{6} \delta_{m,-n} \end{aligned}$$

2.13. In the subsequent sections, we will study in detail the structure theory of SUSY vertex algebras. Here we introduce some basic notation, used in the examples further on.

Let V be a vector superspace over \mathbb{C} . Let z be an even indeterminate and $\theta^1, \dots, \theta^N$ be odd anticommuting indeterminates which commute with z . For an ordered subset $I = (i_1, \dots, i_k) \subset \{1, \dots, N\}$, we will write $\theta^I = \theta^{i_1} \dots \theta^{i_k}$ and let $N \setminus I$ be the ordered complement of I in $\{1, \dots, N\}$.

An $\text{End}(V)$ -valued superfield is an expression of the form:

$$(2.13.1) \quad A(z, \theta^1, \dots, \theta^N) = \sum_{(n|I): n \in \mathbb{Z}} \theta^{N \setminus I} A_{(n|I)} z^{-1-n}$$

where I runs over all ordered subsets of the set $\{1, \dots, N\}$, $A_{(n|I)} \in \text{End}(V)$, and for each I and $v \in V$ we have $A_{(n|I)} v = 0$ for n large enough. We will usually write $A(z, \theta)$ or simply $A(Z)$ for this field, where $Z = (z, \theta^1, \dots, \theta^N)$.

Remark 2.14. Define an $N = 2$ superconformal vertex algebra as a vertex algebra with a vector τ and two operators S^1, S^2 satisfying

$$[T, S^i] = 0, \quad [S^i, S^j] = 2\delta_{i,j}T,$$

such that the corresponding fields $J(z) = -iY(\tau, z)$, $L(z) = \frac{1}{2}Y(S^2S^1\tau, z)$ and

$$(2.14.1) \quad \begin{aligned} G^{(1)}(z) &\equiv G^+(z) + G^-(z) = -Y(S^2\tau, z), \\ G^{(2)}(z) &\equiv i(G^+(z) - G^-(z)) = Y(S^1\tau, z), \end{aligned}$$

satisfy the λ -brackets of Example 2.12, $L_{-1} = T$, $G_{-1/2}^{(i)} = S^i$, and L_0 is diagonalizable with eigenvalues bounded below. Then we obtain an $N_K = 2$ SUSY vertex algebra (see 4.17 for a definition) by letting

$$Y(a, Z) = Y(a, z) + \theta^1 Y(S^1 a, z) + \theta^2 Y(S^2 a, z) + \theta^2 \theta^1 Y(S^1 S^2 a, z).$$

Similarly, given a vertex algebra with two vectors ν, τ and an odd operator S such that $[T, S] = 0$, $S^2 = 0$ and the associated fields:

$$\begin{aligned} J(z) &= -Y(\tau, z), \quad H(z) = Y(\nu, z) \\ Q(z) &= Y(S\tau, z), \quad \tilde{L}(z) = Y(S\nu, z) - \partial_z J(z) \end{aligned}$$

satisfy the commutation relations (2.12.1b), $T_{-1} = T$, $Q_{-1} = S$, T_0 is diagonalizable with eigenvalues bounded below, and J_0 is diagonalizable, we obtain an $N_W = 1$ SUSY vertex algebra (see 3.3.1 for a definition) by letting:

$$Y(a, Z) = Y(a, z) + \theta Y(Sa, z).$$

Example 2.15. Following the previous remark, we can give the $N = 2$ vertex algebra, as defined in Example 2.12, the structure of an $N_K = 2$ SUSY vertex algebra by letting $S^1 = (G_{(0)}^+ + G_{(0)}^-)$ and $S^2 = i(G_{(0)}^+ - G_{(0)}^-)$. Also we check directly that letting

$$(2.15.1) \quad \tau = \sqrt{-1}J_{(-1)}|0\rangle,$$

we get:

$$(2.15.2) \quad Y(\tau, z, \theta^i) = \sqrt{-1}J(z) + \theta^1 G^{(2)}(z) - \theta^2 G^{(1)}(z) + 2\theta^1 \theta^2 L(z)$$

where $G^{(1)}(z) = G^+(z) + G^-(z)$ and $G^{(2)}(z) = i(G^+(z) - G^-(z))$. It follows that $[S^i, S^j] = 2\delta_{ij}T$, $\tau_{(0|0)} = 2T$, $\tau_{(0|1)} = -S^1$ and $\tau_{(0|2)} = -S^2$ (cf. 5.6 below).

We note that $G^{(i)}$ are primary of conformal weight $3/2$, and J is primary of conformal weight 1. The other commutation relations between the generating fields $L, J, G^{(i)}$ ($i = 1, 2$) are

$$\begin{aligned} [G^{(i)}_\lambda G^{(i)}_\lambda] &= 2L + \frac{c\lambda^2}{3}, \quad [G^{(1)}_\lambda G^{(2)}_\lambda] = -i(\partial + 2\lambda)J, \\ [J_\lambda G^{(1)}_\lambda] &= -iG^{(2)}_\lambda, \quad [J_\lambda G^{(2)}_\lambda] = iG^{(1)}_\lambda, \end{aligned}$$

or, equivalently,

$$\begin{aligned} [G_m^{(i)}, G_n^{(i)}] &= 2L_{m+n} + \left(m^2 - \frac{1}{4}\right) \frac{c}{3} \delta_{m,-n}, \quad [G_m^{(1)}, G_n^{(2)}] = i(n-m)J_{m+n}, \\ [J_m, G_n^{(1)}] &= -iG_{m+n}^{(2)}, \quad [J_m, G_n^{(2)}] = iG_{m+n}^{(1)}. \end{aligned}$$

Similarly, we can view the $N = 2$ vertex algebra of Example 2.12 as an $N_W = 1$ SUSY vertex algebra as follows. We will use the generating fields \tilde{L}, Q, H , and J with the commutation relations (2.12.1c). Define the superfields:

$$Y(a, z, \theta) = Y(a, z) + \theta Y(Q_{-1}a, z),$$

and let $T = T_{-1}$, $S = S^1 = Q_{-1}$, so that T and S commute and $S^2 = 0$.

Note that defining the vectors $\nu = H_{(-1)}|0\rangle$ and $\tau = -J_{(-1)}|0\rangle$ we have in particular

$$Y(\nu, z, \theta) = H(z) + \theta(\tilde{L}(z) + \partial_z J(z)),$$

$$Y(\tau, z, \theta) = -J(z) + \theta Q(z).$$

Therefore, if we consider the Fourier modes as defined in (2.13.1), we have

$$\nu_{(0,0)} = T, \quad \tau_{(0,0)} = S.$$

Moreover, it is easy to see that the field $\tilde{L}(z) + \partial_z J(z)$ is also a Virasoro field and the conformal weights of the generating fields \tilde{L}, H, Q, J are positive with respect to this Virasoro field as well. It follows that the operator $\nu_{(1,0)}$ acts diagonally with non-negative eigenvalues (cf. Definition 5.2 below).

Example 2.16. [Kac96, Ex. 5.9d] Consider the vertex algebra generated by a pair of free charged bosons α^\pm and a pair of free charged fermions φ^\pm where the only non-trivial commutation relations are:

$$[\alpha^\pm_\lambda \alpha^\mp] = \lambda, \quad [\varphi^\pm_\lambda \varphi^\mp] = 1.$$

This vertex algebra contains the following family of $N = 2$ vertex subalgebras ($\mu \in \mathbb{C}$):

$$\begin{aligned} S^\pm &=: \alpha^\pm \varphi^\pm : \pm m \partial \varphi^\pm, \quad J =: \varphi^+ \varphi^- : - m(\alpha^+ + \alpha^-), \\ L &=: \alpha^+ \alpha^- : + \frac{1}{2} : \partial \varphi^+ \varphi^- : + \frac{1}{2} : \partial \varphi^- \varphi^+ : - \frac{m}{2} \partial(\alpha^+ - \alpha^-). \end{aligned}$$

The vector τ given by (2.15.1) provides this vertex algebra with the structure of an $N_K = 2$ SUSY vertex algebra, by letting $T = L_{-1}$ and $S^i = G_{-1/2}^{(i)}$ (see (2.14.1)). As in Example 2.15, we can view this vertex algebra as an $N_W = 1$ SUSY vertex algebra.

Example 2.17. An example of an $N_W = N$ SUSY vertex algebra for each N can be constructed as follows. Denote by A the set of all ordered monomials $\theta^{i_1} \dots \theta^{i_s}$ and consider the superalgebra $\mathbb{C}[t, t^{-1}, \theta^1, \dots, \theta^N]$ where t is even and θ^i are odd indeterminates. Let $W(1|N)$ be the Lie superalgebra of derivations of this superalgebra, and define the following collection of $W(1|N)$ -valued formal distributions:

$$\mathcal{F} = \left\{ a^j(z) = \sum_{n \in \mathbb{Z}} (t^n a \partial_j) z^{-1-n} \mid a \in A, j = 0, 1, \dots, N \right\},$$

where $\partial_j = \partial_{\theta^j}$ if $j > 0$ and $\partial_0 = \partial_t$. The pair $(W(1|N), \mathcal{F})$ is a *formal distribution Lie superalgebra*. The corresponding Lie conformal superalgebra is the free $\mathbb{C}[\partial]$ -module \mathcal{W}_N generated by the vectors a^j , with $a \in A$ and $j = 0, \dots, N$, and the following λ -brackets (cf. [Kac96], [FK02]):

$$\begin{aligned} [a^i_\lambda b^j] &= (a \partial_i b)^j + (-1)^{p(a)} ((\partial_j a) b)^i, \quad i, j \geq 1, \\ [a^i_\lambda b^0] &= (a \partial_i b)^0 - (-1)^{p(b)} (a b)^i \lambda, \quad [a^0_\lambda b^0] = -\partial(ab)^0 - 2(ab)^0 \lambda. \end{aligned}$$

Let $V(\mathscr{W}_N)$ be the associated universal enveloping vertex algebra. The field

$$L(z) = -1^0(z) + \sum_{i=1}^n \partial_z(\theta^i)^i(z),$$

is a Virasoro field, and the elements $(\theta^i)^j$ are primary of conformal weight 1, while the elements -1^i are primary of conformal weight 2. We will need later its Fourier modes, which are given by:

$$L_n = -t^{n+1}\partial_t - (n+1) \sum t^n \theta^i \partial_{\theta^i}.$$

We define $T = L_{-1} = -\partial_t$. In order to be consistent with previous notation we define the fields $Q^i(z) = -1^i(z)$ and write down their Fourier modes which are

$$Q_n^i = -t^{n+1}\partial_{\theta^i}.$$

In particular, we define $S^i = Q_{-1}^i$ for $i \geq 1$ and note that $(S^i)^2 = 0$ and $[T, S^i] = 0$.

In order to construct an $N_W = N$ SUSY vertex algebra from the vertex algebra $V(\mathscr{W}_N)$ we proceed as before, defining the superfields

$$(2.17.1) \quad Y(a, z, \theta^1, \dots, \theta^N) = \sum_I (-1)^{\frac{I(I-1)}{2}} \theta^I Y(S^{i_1} \dots S^{i_s} a, z),$$

where the summation is taken over all ordered subsets $I = (i_1, \dots, i_s)$ of $\{1, \dots, N\}$. It is straightforward to check that the data $(V(\mathscr{W}_N), T, S^i, |0\rangle, Y)$ is indeed an $N_W = N$ SUSY vertex algebra. Moreover, this is a $W(1|N)$ -conformal $N_W = N$ SUSY vertex algebra, as defined in 1.5. We shall return to this example in 5.1.

Example 2.18. We can similarly construct an $N_K = N$ SUSY vertex algebra $V(\mathscr{K}_N)$ for any N . For this we define a subalgebra $K(1|N)$ of $W(1|N)$, of those differential operators preserving the form $\omega = dt + \sum \theta^i d\theta^i$ up to multiplication by a function (recall that we consider d to be an even derivation, as in [DM99], but not in [KvdL89] and [FK02]). It consists of differential operators of the form:

$$(2.18.1) \quad D^f = f\partial_0 + \frac{1}{2}(-1)^{p(f)} \sum_{i=1}^N (\theta^i \partial_0 + \partial_i)(f)(\theta_i \partial_0 + \partial_i),$$

for $f \in \mathbb{C}[t, t^{-1}, \theta^1, \dots, \theta^N]$. These operators satisfy

$$[D^f, D^g] = D^{\{f, g\}},$$

where

$$\{f, g\} = \left(f - \frac{1}{2} \sum_{i=1}^N \theta^i \partial_i f \right) \partial_0 g - \partial_0 f \left(g - \frac{1}{2} \sum_{i=1}^N \theta^i \partial_i g \right) + (-1)^f \frac{1}{2} \sum_{i=1}^N \partial_i f \partial_i g.$$

In particular $K(1|N)$ contains the operators

$$L_n = -t^{n+1}\partial_t - \frac{n+1}{2}t^n \sum \theta^i \partial_{\theta^i}, \quad n \in \mathbb{Z},$$

$$G_n^{(i)} = -t^{n+1/2}(\partial_{\theta^i} - \theta^i \partial_t) + \left(n + \frac{1}{2} \right) t^{n-1/2} \theta^i \sum \theta^j \partial_{\theta^j}, \quad n \in \frac{1}{2} + \mathbb{Z}.$$

It is easy to see that the operators L_n span a centerless Virasoro Lie algebra.

As in the $W(1|N)$ case, we construct the corresponding Lie conformal superalgebra as follows. It is the free $\mathbb{C}[\partial]$ -module \mathcal{K}_N generated by vectors $a \in A$, with the following λ -brackets [FK02]

$$[a_\lambda b] = \left(\left(\frac{r}{2} - 1 \right) \partial(ab) + (-1)^r \frac{1}{2} \sum_{i=1}^n \partial_i a \partial_i b \right) + \lambda \left(\frac{r+s}{2} - 2 \right) ab,$$

where $a = \theta^{i_1} \dots \theta^{i_r}$, $b = \theta^{j_1} \dots \theta^{j_s}$.

We denote by $V(\mathcal{K}_N)$ its universal enveloping vertex algebra, and we define the operators $T = L_{-1}$ and $S^i = G_{-1/2}^{(i)}$. Now we define the state-field correspondence as in (2.17.1):

$$Y(a, z, \theta) = \sum_I (-1)^{\frac{I(I-1)}{2}} \theta^I Y(S^I a, z).$$

All the properties of an $N_K = N$ SUSY vertex algebra are straightforward to check as in the previous cases. Moreover, this is a $K(1|N)$ -conformal $N_K = N$ SUSY vertex algebra. We will return to this example in 5.5.

3. STRUCTURE THEORY OF $N_W = N$ SUSY VERTEX ALGEBRAS

In this section we develop the structure theory of SUSY Lie conformal algebras and SUSY vertex algebras along the lines of [Kac96] (see also [DSK05] for a better exposition). Proofs are rather straightforward adaptations of those in the vertex algebra case, the only difficulty being the problem of signs.

3.1. Formal distribution calculus.

3.1.1. In what follows we fix the ground field to be the complex numbers \mathbb{C} and N to be a non-negative integer. Let $\theta^1, \dots, \theta^N$ be Grassmann variables and $I = \{i_1, \dots, i_k\}$ be an *ordered k -tuple*: $1 \leq i_1 < \dots < i_k \leq N$. We will denote

$$\theta^I = \theta^{i_1} \dots \theta^{i_k}, \quad \theta^N = \theta^1 \dots \theta^N.$$

For an element a in a vector superspace we will denote $(-1)^a = (-1)^{p(a)}$, where $p(a) \in \mathbb{Z}/2\mathbb{Z}$ is the parity of a , and, given a k -tuple I as above, we will let $(-1)^I = (-1)^k$. Given two disjoint ordered tuples I and J , we define $\sigma(I, J) = \pm 1$ by

$$\theta^I \theta^J = \sigma(I, J) \theta^{I \cup J},$$

and we define $\sigma(I, J)$ to be zero if $I \cap J \neq \emptyset$. Also, unless noted otherwise, all “union” symbols “ \cup ” will denote disjoint unions³. It follows easily, by looking at $\theta^I \theta^J \theta^K$, that for three mutually disjoint tuples, I , J and K we have:

$$(3.1.1.1) \quad \sigma(I, J) \sigma(I \cup J, K) = \sigma(I, J \cup K) \sigma(J, K), \quad \sigma(I, J) = (-1)^{IJ} \sigma(J, I).$$

Here and further $(-1)^{IJ}$ stands for $(-1)^{(\#I)(\#J)}$.

We will denote by $N \setminus I$ the ordered complement of I in $\{1, \dots, N\}$ and define $\sigma(I) := \sigma(I, N \setminus I)$. It follows from the definitions that $\theta^I \theta^{N \setminus I} = \sigma(I) \theta^N$.

3.1.2. Let $Z = (z, \theta^1, \dots, \theta^N)$ and $W = (w, \zeta^1, \dots, \zeta^N)$ denote two sets of coordinates in the formal superdisk $D = D^{1|N}$. As before, all θ^i and ζ^j anticommute.

Let $\mathbb{C}[[z]]$ be the algebra of formal power series in z ; its elements are series $\sum_{n \geq 0} a_n z^n$ with $a_n \in \mathbb{C}$. The superalgebra of regular functions in D is defined as

³This will not be true in section 4 where we analyze $N_K = n$ SUSY vertex algebras

$\mathbb{C}[[Z]] := \mathbb{C}[[z]] \otimes \mathbb{C}[\theta^1, \dots, \theta^N]$. Similarly, we define the superalgebra $\mathbb{C}[[Z, W]] := \mathbb{C}[[z, w]] \otimes \mathbb{C}[\theta^1, \dots, \theta^N, \zeta^1, \dots, \zeta^N]$.

For any \mathbb{C} -algebra R , we denote by $R((z))$ the algebra of R -valued formal Laurent series, its elements are series of the form $\sum_{n \in \mathbb{Z}} a_n z^n$ such that $a_n \in R$ and there exists $N_0 \in \mathbb{Z}$ such that $a_n = 0$ for all $n \leq N_0$. If R is a field, so is $R((z))$. We denote $R((Z)) := R((z)) \otimes_{\mathbb{C}} \mathbb{C}[\theta^1, \dots, \theta^N]$. Denote also by $\mathbb{C}((Z))((W))$ the superalgebra $R((W))$ where $R = \mathbb{C}((Z))$; its elements are Laurent series in W whose coefficients are Laurent series in Z . Similarly we have the superalgebra $\mathbb{C}((W))((Z))$.

Denote by $\mathbb{C}((z, w))$ the field of fractions of $\mathbb{C}[[z, w]]$ and put $\mathbb{C}((Z, W)) := \mathbb{C}((z, w)) \otimes_{\mathbb{C}} \mathbb{C}[\theta^1, \dots, \theta^N, \zeta^1, \dots, \zeta^N]$. One may think of this superalgebra as the algebra of meromorphic functions in the formal superdisk $D^{2|2N}$. Given such a meromorphic function, we can “expand it near the w axis”, to obtain an element of $\mathbb{C}((Z))((W))$. Indeed, $\mathbb{C}[[z, w]]$ embeds naturally in $\mathbb{C}((z))((w))$ and $\mathbb{C}((w))((z))$ respectively. Since $\mathbb{C}((z, w))$ is the ring of fractions of $\mathbb{C}[[z, w]]$ and $\mathbb{C}((z))((w))$ and $\mathbb{C}((w))((z))$ are fields, these embeddings induce respective algebra embeddings

$$\mathbb{C}((z))((w)) \xleftrightarrow{i_{z,w}} \mathbb{C}((z, w)) \xleftrightarrow{i_{w,z}} \mathbb{C}((w))((z)).$$

(A concrete example is given by (2.4.1)).

Tensoring with the corresponding Grassmann superalgebras, we obtain superalgebra embeddings

$$\mathbb{C}((Z))((W)) \xleftrightarrow{i_{z,w}} \mathbb{C}((Z, W)) \xleftrightarrow{i_{w,z}} \mathbb{C}((W))((Z)).$$

Let \mathcal{U} be a vector superspace. An \mathcal{U} -valued *formal distribution* is an expression of the form

$$a(Z) = \sum_{(n|I): n \in \mathbb{Z}} Z^{n|I} a_{n|I}, \quad a_{n|I} \in \mathcal{U}.$$

The space of such distributions will be denoted $\mathcal{U}[[Z, Z^{-1}]]$. We denote by $\mathbb{C}[Z, Z^{-1}] := \mathbb{C}[z, z^{-1}] \otimes \mathbb{C}[\theta^1, \dots, \theta^N]$ the superalgebra of Laurent polynomials. A \mathcal{U} -valued formal distribution is canonically a linear functional $\mathbb{C}[Z, Z^{-1}] \rightarrow \mathcal{U}$. To see this, we define the *super residue* as the coefficient of $Z^{-1|N}$:

$$\text{res}_Z a(Z) = a_{-1|N}.$$

This clearly satisfies

$$(3.1.2.1) \quad \text{res}_Z \partial_z a(Z) = \text{res}_Z \partial_\theta a(Z) = 0.$$

Given a \mathcal{U} -valued formal distribution $a(Z)$ we obtain a linear map $\mathbb{C}[Z, Z^{-1}] \rightarrow \mathcal{U}$ given by

$$f(Z) \mapsto \text{res}_Z a(Z) f(Z).$$

Conversely, every formal distribution arises in this way. Indeed we have:

$$(3.1.2.2) \quad \text{res}_Z Z^{n|I} a(Z) = \sigma(I) a_{-1-n|N \setminus I}.$$

Therefore the formal distribution $a(Z)$ can be written as

$$a(Z) = \sum_{(n|I): n \in \mathbb{Z}} Z^{-1-n|N \setminus I} a_{(n|I)},$$

where

$$a_{(n|I)} = \sigma(I) \text{res}_Z Z^{n|I} a(Z).$$

We can similarly define \mathcal{U} -valued formal distributions in two variables, as expressions of the form

$$a(Z, W) = \sum_{(j|J), (k|K)} Z^{j|J} W^{k|K} a_{j|J, k|K}, \quad a_{j|J, k|K} \in \mathcal{U}.$$

The space of such formal distributions will be denoted $\mathcal{U}[[Z, Z^{-1}, W, W^{-1}]]$.

Note that in the case $\mathcal{U} = \mathbb{C}$, both algebras $\mathbb{C}((Z))((W))$ and $\mathbb{C}((W))((Z))$ are embedded in $\mathbb{C}[[Z, Z^{-1}, W, W^{-1}]]$. We will denote by $i_{z,w}$ and $i_{w,z}$ the corresponding embeddings of $\mathbb{C}((Z, W))$ in $\mathbb{C}[[Z, Z^{-1}, W, W^{-1}]]$. When $f(Z, W)$ is a Laurent polynomial (that is a polynomial in z, z^{-1}, w, w^{-1} and the odd variables) then the embeddings $i_{z,w}f$ and $i_{w,z}f$ coincide on $\mathbb{C}[[Z, Z^{-1}, W, W^{-1}]]$. Indeed, it is immediate to see that

$$(3.1.2.3) \quad \mathbb{C}((Z))((W)) \cap \mathbb{C}((W))((Z)) = \mathbb{C}[[Z, W]][z^{-1}, w^{-1}],$$

where the intersection is taken in $\mathbb{C}[[Z, Z^{-1}, W, W^{-1}]]$. The images under these embeddings are different for other functions, as we will see below (cf. 3.1.5).

A \mathcal{U} -valued formal distribution in two variables is called *local* if there exists a non-negative integer n such that

$$(z - w)^n a(Z, W) = 0.$$

3.1.3. Note that the differential operators $\partial_z, \partial_{\theta^i}$ and $\partial_w, \partial_{\zeta^i}$ act in the usual way on the spaces $\mathbb{C}((Z, W))$, $\mathbb{C}[[Z, Z^{-1}, W, W^{-1}]]$. For $j \in \mathbb{Z}_+$ and $J = (j_1, \dots, j_k)$ we will denote

$$\partial_Z^{j|J} = \partial_z^j \partial_{\theta^{j_1}} \dots \partial_{\theta^{j_k}}.$$

We define

$$\partial_Z^{(j|J)} := \frac{(-1)^{\frac{J(J+1)}{2}}}{j!} \partial_Z^{j|J}, \quad Z^{(j|J)} := \frac{(-1)^{\frac{J(J+1)}{2}}}{j!} Z^{j|J}.$$

One checks easily that the embeddings $i_{z,w}$ and $i_{w,z}$ defined above, commute with the action of the differential operators $\partial_Z^{j|J}$ and $\partial_W^{j|J}$.

We will denote

$$(3.1.3.1) \quad \begin{aligned} Z - W &= (z - w, \theta^1 - \zeta^1, \dots, \theta^N - \zeta^N), & Z^{n|I} &= z^n \theta^I, \\ (Z - W)^{j|J} &= (z - w)^j \prod_{i \in J} (\theta^i - \zeta^i), & \partial_W &= (\partial_w, \partial_{\zeta^1}, \dots, \partial_{\zeta^N}), \end{aligned}$$

and for any \mathcal{U} -valued formal distribution $f(Z)$, we define its *Taylor expansion* as:

$$(3.1.3.2) \quad f(Z) = e^{(Z-W)\partial_W} f(W),$$

where

$$(Z - W)\partial_W = (z - w)\partial_w + \sum_i (\theta^i - \zeta^i)\partial_{\zeta^i}.$$

Expanding the exponential in (3.1.3.2) we obtain:

$$f(Z) = \sum_{(j|J): j \geq 0} (-1)^J (Z - W)^{j|J} \partial_W^{(j|J)} f(W).$$

Remark 3.1.4. In the definition of formal distributions and super residues, we can replace \mathbb{C} by any commutative superalgebra \mathcal{A} , and \mathcal{U} by any \mathcal{A} -module. We see immediately that the residue map is of parity $N \bmod 2$, that is, for $\chi \in \mathcal{A}$, and $u(Z)$ an \mathcal{U} -valued distribution, we have:

$$\text{res}_Z \chi u(Z) = (-1)^{\chi^N} \text{res}_Z u(Z).$$

On the other hand, this residue map is a morphism of right \mathcal{A} -modules, namely:

$$\text{res}_Z u(Z) \chi = \left(\text{res}_Z u(Z) \right) \chi.$$

Proposition 3.1.5. *There exists a unique \mathbb{C} -valued formal distribution $\delta(Z, W)$ such that for every function $f \in \mathcal{U}[ZZ^{-1}]$ we have $\text{res}_Z \delta(Z, W) f(Z) = f(W)$.*

Proof. For uniqueness, let δ and δ' be two such distributions, then $\beta = \delta - \delta'$ satisfies $\text{res}_Z \beta(Z, W) f(Z) = 0$ for all functions $f(Z)$. Decomposing $\beta(Z, W) = \sum \beta_n |I, m|_J W^{m|J} Z^{n|I}$, and multiplying by $Z^{k|L}$ we see that $\beta_{-1-k|N-L, m|J} = 0$ for all $m|J$, hence $\beta = 0$. Existence will be proved below. \square

3.1.6. We define the formal δ -function as the \mathbb{C} -valued formal distribution in two variables, given by

$$(3.1.6.1) \quad \delta(Z, W) = (i_{z,w} - i_{w,z})(Z - W)^{-1|N} = (i_{z,w} - i_{w,z}) \frac{(\theta - \zeta)^N}{z - w}$$

It follows that

$$\partial_w^{(n)} \delta(Z, W) := \frac{1}{n!} \partial_w^n \delta(Z, W) = (i_{z,w} - i_{w,z})(Z - W)^{-1-n|N}.$$

This distribution has the following properties:

- (1) $(Z - W)^{m|J} \partial_W^{n|I} \delta(Z, W) = 0$ whenever $m > n$ or $J \not\supseteq I$,
- (2) $(Z - W)^{j|J} \partial_W^{(n|I)} \delta(Z, W) = \sigma(I \setminus J, J) \partial_W^{(n-j|I \setminus J)} \delta(Z, W)$ if $n \geq j$ and $I \supset J$,
- (3) $\delta(Z, W) = (-1)^N \delta(W, Z)$,
- (4) $\partial_Z^{j|J} \delta(Z, W) = (-1)^{j+N+J} \partial_W^{j|J} \delta(W, Z)$,
- (5) $\delta(Z, W) a(Z) = \delta(Z, W) a(W)$, where $a(Z)$ is any formal distribution,
- (6) $\text{res}_Z \delta(Z, W) a(Z) = a(W)$, where $a(Z)$ is any formal distribution,
- (7) $\exp((Z - W)\Lambda) \partial_W^{n|I} \delta(Z, W) = (\Lambda + \partial_W)^{n|I} \delta(Z, W)$, where $\Lambda = (\lambda, \chi^1, \dots, \chi^N)$, χ^i are odd anticommuting variables, λ is even, λ commutes with χ^i , and we write

$$(Z - W)\Lambda = (z - w)\lambda + \sum_i (\theta^i - \zeta^i) \chi^i, \quad (\Lambda + \partial_W) = (\lambda + \partial_w, \chi_i + \partial_{\theta^i}).$$

Proof. Writing $\partial_\zeta^I = \partial_{\zeta^{i_1}} \dots \partial_{\zeta^{i_k}}$ we have

$$(Z - W)^{m|J} \partial_W^{n|I} \delta(Z, W) = (z - w)^m n! (i_{z,w} - i_{w,z})(z - w)^{-1-n} (\theta - \zeta)^J \partial_\zeta^I (\theta - \zeta)^N$$

Now this clearly vanishes if $m \geq 1 + n$ since then the two embeddings $i_{z,w}$ and $i_{w,z}$ coincide on the regular function $(z - w)^{m-n-1}$. The other factor is clearly zero if $J \not\supseteq I$ since for every $j \in J \setminus I$ we have a factor $(\theta^j - \zeta^j)$ in $\partial_\zeta^I (\theta - \zeta)^N$. This proves (1).

In order to prove (2) we write:

$$\begin{aligned}
(3.1.6.2) \quad & (Z - W)^{j|J} \partial_W^{n|I} \delta(Z, W) = n! (i_{z,w} - i_{w,z})(z - w)^{j-1-n} (\theta - \zeta)^J \partial_\zeta^I (\theta - \zeta)^N \\
& = \frac{n!}{(n-j)!} (n-j)! (i_{z,w} - i_{w,z})(z - w)^{-1-(n-j)} (-1)^J \sigma(J, I \setminus J) (\theta - \zeta)^J \partial_\theta^J \partial_\zeta^{I \setminus J} (\theta - \zeta)^N \\
& = \frac{n!}{(n-j)!} (-1)^{\frac{J(J+1)}{2}} \sigma(J, I \setminus J) (n-j)! (i_{z,w} - i_{w,z})(z - w)^{-1-(n-j)} \partial_\zeta^{I \setminus J} (\theta - \zeta)^N \\
& = (-1)^{\frac{J(J+1)}{2}} \sigma(J, I \setminus J) \frac{n!}{(n-j)!} \partial_W^{n-j|I \setminus J} \delta(Z, W).
\end{aligned}$$

(3) is obvious and (4) follows from (3) easily. In order to prove (5) we see that from (1) we have $\delta z = \delta w$ therefore we get $\delta(Z, W) z^k = \delta(Z, W) w^k$. On the other hand, also from (1) it follows that $\delta(Z, W) \theta^i = \delta(Z, W) \zeta^i$. Hence $\delta(Z, W) \theta^I = \delta(Z, W) \zeta^I$ and we have proved that $\delta(Z, W) Z^{n|I} = \delta(Z, W) W^{n|I}$. The result follows by linearity now.

(6) follows by taking residue in (5). To prove (7) we first expand the exponential in power series:

$$(3.1.6.3) \quad \exp((Z - W)\Lambda) \partial_W^{n|I} \delta(Z, W) = \sum_{(j|J): j \geq 0} (-1)^J (Z - W)^{(j|J)} \Lambda^{j|J} \partial_W^{n|I} \delta(Z, W).$$

Now using (2) we see that this is:

$$(3.1.6.4) \quad \sum_{(j|J): j \geq 0} \binom{n}{j} \Lambda^{j|J} \sigma(J, I \setminus J) \partial_W^{n-j|I \setminus J} \delta(Z, W).$$

On the other hand we can expand the right hand side of (7) as:

$$\begin{aligned}
(\Lambda + \partial_W)^{n|I} &= (\lambda + \partial_w)^n (\chi + \partial_\zeta)^I = \sum_{(j|J): j \geq 0} \binom{n}{j} \lambda^j \partial_w^{n-j} \sigma(J, I \setminus J) \chi^J \partial_\zeta^{I \setminus J} \\
&= \sum_{(j|J): j \geq 0} \binom{n}{j} \Lambda^{j|J} \sigma(J, I \setminus J) \partial_W^{n-j|I \setminus J}.
\end{aligned}$$

Comparing with (3.1.6.4) we get the result. \square

Lemma 3.1.7. *Let $a(Z, W)$ be a local formal distribution in two variables. Then $a(Z, W)$ can be uniquely decomposed as*

$$(3.1.7.1) \quad a(Z, W) = \sum_{(j|J): j \geq 0} \left(\partial_W^{(j|J)} \delta(Z, W) \right) c_{j|J}(W),$$

where the sum is finite. The coefficients $c_{j|J}$ are given by

$$(3.1.7.2) \quad c_{j|J}(W) = \text{res}_Z (Z - W)^{j|J} a(Z, W).$$

Proof. First we note that if $a(Z, W)$ is local then the sum on the right hand side is finite. Let $b(Z, W)$ the difference between the right hand side and the left hand

side of (3.1.7.1). We find:

$$\begin{aligned}
\operatorname{res}_Z(Z-W)^{k|K}b(Z,W) &= \operatorname{res}_Z(Z-W)^{k|K}a(Z,W) - \\
&\quad - \operatorname{res}_Z \sum_{(j|J):j \geq 0} (Z-W)^{k|K} \left(\partial_W^{(j|J)} \delta(Z,W) \right) c_{j|J}(W) \\
&= c_{k|K}(W) - \operatorname{res}_Z \left(\partial_W^{(j-k|J \setminus K)} \delta(Z,W) \right) c_{(j|J)}(W) \\
&= c_{k|K}(W) - \operatorname{res}_Z \delta(Z,W) c_{k|K}(W) = 0,
\end{aligned}$$

where in the second line we have used (2) of 3.1.6. It follows that $b(Z,W)$ has no negative powers of z . Moreover, $b(Z,W)$ is local, since $a(Z,W)$ is, and the right hand side of (3.1.7.1) is local by (1) of 3.1.6. We can write then

$$b(Z,W) = \sum_{(j|J):j \geq 0} Z^{j|J} b_{j|J}(W),$$

and since $(z-w)^n b(Z,W) = 0$ we obtain:

$$\sum_{\substack{(j|J) \\ j \geq k \geq 0}} \binom{n}{k} Z^{j|J} w^{n-k} b_{j-k|J}(W) = 0,$$

which easily shows that $b(Z,W) = 0$. Uniqueness is clear by taking residues on both sides of (3.1.7.1). \square

3.1.8. Let $a(Z,W)$ be a formal distribution in two variables. We define its formal Fourier transform by:

$$(3.1.8.1) \quad \mathcal{F}_{Z,W}^\Lambda a(Z,W) = \operatorname{res}_Z \exp((Z-W)\Lambda) a(Z,W),$$

where $\Lambda = (\lambda, \chi^1, \dots, \chi^N)$, λ is an even variable, and χ^i are odd anticommuting variables, commuting with λ .

Expanding this exponential we have (recall (3.1.6.3)) :

$$\begin{aligned}
(3.1.8.2) \quad \mathcal{F}_{Z,W}^\Lambda a(Z,W) &= \operatorname{res}_Z \sum_{(j|J):j \geq 0} \Lambda^{j|J} (Z-W)^{(j|J)} a(Z,W) \\
&= \sum_{(j|J):j \geq 0} (-1)^{JN} \Lambda^{j|J} \operatorname{res}_Z (Z-W)^{(j|J)} a(Z,W) = \sum_{(j|J):j \geq 0} (-1)^{JN} \Lambda^{(j|J)} c_{j|J}(W)
\end{aligned}$$

where $c_{j|J}$ are defined by (3.1.7.2) and we write, as before

$$\Lambda^{j|J} = \lambda^j \chi^{j_1} \dots \chi^{j_k}, \quad \Lambda^{(j|J)} := \frac{(-1)^{\frac{J(J+1)}{2}}}{j!} \Lambda^{j|J}.$$

Proposition 3.1.9. *The formal Fourier transform satisfies the following properties:*

(1) *sesquilinearity:*

$$\begin{aligned}
\mathcal{F}_{Z,W}^\Lambda \partial_z a(Z,W) &= -\lambda \mathcal{F}_{Z,W}^\Lambda a(Z,W) = [\partial_w, \mathcal{F}_{Z,W}^\Lambda] a(Z,W), \\
\mathcal{F}_{Z,W}^\Lambda \partial_{\theta^i} a(Z,W) &= -(-1)^N \chi^i \mathcal{F}_{Z,W}^\Lambda a(Z,W) = (-1)^N [\partial_{\zeta^i}, \mathcal{F}_{Z,W}^\Lambda] a(Z,W).
\end{aligned}$$

(2) For any local formal distribution $a(Z, W)$ we have:

$$(3.1.9.1) \quad \begin{aligned} (-1)^N \mathcal{F}_{Z,W}^\Lambda a(W, Z) &= \mathcal{F}_{Z,W}^{-\Lambda - \partial_W} a(Z, W), \\ &= \mathcal{F}_{Z,W}^\Gamma a(Z, W)|_{\Gamma = -\Lambda - \partial_W}, \end{aligned}$$

where $-\Lambda - \partial_W = (-\lambda - \partial_w, -\chi^i - \partial_{\zeta^i})$.

(3) For any formal distribution in three variables $a(Z, W, X)$ we have

$$\mathcal{F}_{Z,W}^\Lambda \mathcal{F}_{X,W}^\Gamma a(Z, W, X) = (-1)^N \mathcal{F}_{X,W}^{\Lambda+\Gamma} \mathcal{F}_{Z,X}^\Lambda a(Z, W, X),$$

where $\Gamma = (\gamma, \eta^1, \dots, \eta^N)$, with η^i odd anticommutative variables and γ is even and commutes with η^i , $\Lambda + \Gamma$ is the sum $(\lambda + \gamma, \chi^i + \eta^i)$, and the superalgebra $\mathbb{C}[\Lambda, \Gamma]$ is commutative.

Proof. The proof of the first equality of both lines of (1) follows from (3.1.2.1). For the first equality of the second line we have

$$\begin{aligned} \mathcal{F}_{Z,W}^\Lambda \partial_{\theta^i} a(Z, W) &= \text{res}_Z \exp((Z - W)\Lambda) \partial_{\theta^i} a(Z, W) \\ &= -\text{res}_Z (\partial_{\theta^i} \exp((Z - W)\Lambda)) a(Z, W) = -\text{res}_Z \chi^i \exp((Z - W)\Lambda) a(Z, W) \\ &= -(-1)^N \chi^i \mathcal{F}_{Z,W}^\Lambda a(Z, W). \end{aligned}$$

For the second equality of the second line of (1) we have:

$$\begin{aligned} [\partial_{\zeta^i}, \mathcal{F}_{Z,W}^\Lambda] a(Z, W) &= (-1)^N (\text{res}_Z \partial_{\zeta^i} \exp((Z - W)\Lambda) a(Z, W) - \\ &\quad - \exp((Z - W)\Lambda) \partial_{\zeta^i} a(Z, W)) = (-1)^N \text{res}_Z (\partial_{\zeta^i} \exp((Z - W)\Lambda)) a(Z, W) \\ &= -\chi^i \mathcal{F}_{Z,W}^\Lambda a(Z, W). \end{aligned}$$

To prove (2) it is enough, by Lemma 3.1.7, to prove the statement when $a(Z, W) = (\partial_W^{j|J} \delta(Z, W)) c(W)$. In this case we have:

$$\mathcal{F}_{Z,W}^\Lambda a(W, Z) = \mathcal{F}_{Z,W}^\Lambda (\partial_Z^{j|J} \delta(W, Z)) c(Z) = \mathcal{F}_{Z,W}^\Lambda (-1)^{j+J+N} \partial_W^{j|J} \delta(Z, W) c(Z).$$

Now using (7) in 3.1.6 we can express the last expression above as:

$$\begin{aligned} &(-1)^{j+J+N} \text{res}_Z (\Lambda + \partial_W)^{j|J} \delta(Z, W) c(Z) = \\ &= (-1)^{j+J+JN+N} (\Lambda + \partial_W)^{j|J} \text{res}_Z \delta(Z, W) c(Z) = (-1)^{j+J+JN+N} (\Lambda + \partial_W)^{j|J} c(W). \end{aligned}$$

On the other hand we have

$$\begin{aligned} \mathcal{F}_{Z,W}^\Gamma a(Z, W)|_{\Gamma = -\Lambda - \partial_W} &= (-1)^{JN} (-\Lambda - \partial_W)^{j|J} c(W) \\ &= (-1)^{j+J+JN} (\Lambda + \partial_W)^{j|J} c(W). \end{aligned}$$

The proof of (3) is straightforward:

$$\begin{aligned} (3.1.9.2) \quad \mathcal{F}_{Z,W}^\Lambda \mathcal{F}_{X,W}^\Gamma &= \text{res}_Z \exp((Z - W)\Lambda) \text{res}_X \exp((X - W)\Gamma) = \\ &= \text{res}_Z \text{res}_X \exp((Z - W)\Lambda + (X - W)\Gamma) = \\ &= (-1)^N \text{res}_X \text{res}_Z \exp((Z - X)\Lambda + (X - W)(\Lambda + \Gamma)) = \\ &= (-1)^N \text{res}_X \exp((X - W)(\Lambda + \Gamma)) \text{res}_Z \exp((Z - X)\Lambda) = (-1)^N \mathcal{F}_{X,W}^{\Lambda+\Gamma} \mathcal{F}_{Z,X}^\Lambda. \end{aligned}$$

The sign $(-1)^N$ appears when we commute the residue maps (recall that they have parity $N \bmod 2$). \square

3.2. $N_W = N$ SUSY Lie conformal algebras.

3.2.1. Let \mathfrak{g} be a Lie superalgebra. A pair of \mathfrak{g} -valued formal distributions $a(Z), b(Z)$ is called *local* if the distribution $[a(Z), b(W)]$ is local. By the decomposition Lemma 3.1.7 we have for such a pair:

$$[a(Z), b(W)] = \sum_{(j|J): j \geq 0} \left(\partial_W^{(j|J)} \delta(Z, W) \right) c_{j|J}(W).$$

where

$$c_{j|J}(W) = \text{res}_Z(Z - W)^{j|J} [a(Z), b(W)].$$

We define $a(W)_{(j|J)} b(W) = c_{j|J}(W)$ and we call this operation the $(j|J)$ -product. Let us also define the Λ -bracket of two \mathfrak{g} -valued formal distributions by

$$(3.2.1.1) \quad [a_\Lambda b](W) = \mathcal{F}_{Z,W}^\Lambda [a(Z), b(W)],$$

where $\mathcal{F}_{Z,W}^\Lambda$ is the formal Fourier transform defined in 3.1.8. It follows from the definitions and from (3.1.8.2) that

$$[a_\Lambda b] = \sum_{(j|J): j \geq 0} (-1)^{JN} \Lambda^{(j|J)} a_{(j|J)} b.$$

Note also that the Λ -bracket has parity $N \bmod 2$ (this follows from the fact that the residue map has parity $N \bmod 2$).

A pair $(\mathfrak{g}, \mathcal{R})$ consisting of a Lie superalgebra \mathfrak{g} and a family \mathcal{R} of pairwise local \mathfrak{g} -valued formal distributions $a(Z)$, whose coefficients span \mathfrak{g} , stable under all $j|J$ -th products and under the derivations ∂_z and ∂_{θ^i} is called an $N_W = N$ *formal distribution Lie superalgebra*.

Proposition 3.2.2. *The Λ -bracket defined in (3.2.1.1) satisfies the following properties:*

(1) *Sesquilinearity for a pair $(a(Z), b(W))$:*

$$(3.2.2.1) \quad [\partial_z a_\Lambda b] = -\lambda [a_\Lambda b] \quad [a_\Lambda \partial_w b] = (\partial_w + \lambda) [a_\Lambda b]$$

$$(3.2.2.2) \quad [\partial_{\theta^i} a_\Lambda b] = -(-1)^N \chi^i [a_\Lambda b] \quad [a_\Lambda \partial_{\zeta^i} b] = (-1)^{a+N} (\partial_{\zeta^i} + \chi^i) [a_\Lambda b]$$

(2) *Skew-symmetry for a local pair $(a(Z), b(W))$:*

$$[b_\Lambda a] = -(-1)^{ab+N} [a_{-\Lambda - \partial_w} b].$$

(3) *Jacobi identity for a triple $(a(Z), b(X), c(W))$:*

$$[a_\Lambda [b_\Gamma c]] = (-1)^{aN+N} [[a_\Lambda b]_{\Lambda+\Gamma} c] + (-1)^{(a+N)(b+N)} [b_\Gamma [a_\Lambda c]],$$

where $\Gamma = (\gamma, \eta^1, \dots, \eta^N)$ and the superalgebra $\mathbb{C}[\Lambda, \Gamma]$ is commutative.

Proof. In order to prove the first equation in (3.2.2.2) we use Proposition 3.1.9 (1):

$$\begin{aligned} [\partial_{\theta^i} a_\Lambda b] &= \mathcal{F}_{Z,W}^\Lambda [\partial_{\theta^i} a(Z), b(W)] = \mathcal{F}_{Z,W}^\Lambda \partial_{\theta^i} [a(Z), b(W)] \\ &= -(-1)^N \chi^i \mathcal{F}_{Z,W}^\Lambda [a(Z), b(W)] = -(-1)^N \chi^i [a_\Lambda b]. \end{aligned}$$

For the second equation we have by Proposition 3.1.9 (1):

$$\begin{aligned} [a_\Lambda \partial_{\zeta^i} b] &= \mathcal{F}_{Z,W}^\Lambda [a(Z), \partial_{\zeta^i} b(W)] = \mathcal{F}_{Z,W}^\Lambda (-1)^a \partial_{\zeta^i} [a(Z), b(W)] \\ &= (-1)^a ([\mathcal{F}_{Z,W}^\Lambda, \partial_{\zeta^i}] + (-1)^N \partial_{\zeta^i} \mathcal{F}_{Z,W}^\Lambda) [a(Z), b(W)] \\ &= (-1)^{a+N} (\chi^i + \partial_{\zeta^i}) \mathcal{F}_{Z,W}^\Lambda [a(Z), b(W)] = (-1)^{a+N} (\chi^i + \partial_{\zeta^i}) [a_\Lambda b]. \end{aligned}$$

Skew-symmetry follows from the skew-symmetry property of the Fourier transform (3.1.9.1) as follows:

$$\begin{aligned} [b_\Lambda a] &= \mathcal{F}_{Z,W}^\Lambda [b(Z), a(W)] = -(-1)^{ab} \mathcal{F}_{Z,W}^\Lambda [a(W), b(Z)] \\ &= -(-1)^{ab+N} \mathcal{F}_{Z,W}^{-\Lambda-\partial_W} [a(Z), b(W)] = -(-1)^{ab+N} [a_{-\Lambda-\partial_W} b]. \end{aligned}$$

Finally, to prove the Jacobi identity we use Proposition 3.1.9 (3)

$$\begin{aligned} [a_\Lambda [b_\Gamma c]] &= \mathcal{F}_{Z,W}^\Lambda [a(Z), \mathcal{F}_{X,W}^\Gamma [b(X), c(W)]] \\ &= (-1)^{aN} \mathcal{F}_{Z,W}^\Lambda \mathcal{F}_{X,W}^\Gamma [a(Z), [b(X), c(W)]] \\ &= (-1)^{aN} \mathcal{F}_{Z,W}^\Lambda \mathcal{F}_{X,W}^\Gamma [[a(Z), b(X)], c(W)] + \\ &\quad + (-1)^{ab+aN} \mathcal{F}_{Z,W}^\Lambda \mathcal{F}_{X,W}^\Gamma [b(X), [a(Z), c(W)]] \\ &= (-1)^{aN+N} \mathcal{F}_{X,W}^{\Lambda+\Gamma} \mathcal{F}_{Z,X}^\Lambda [[a(Z), b(X)], c(W)] + \\ &\quad + (-1)^{ab+aN+bN+N} \mathcal{F}_{X,W}^\Gamma \mathcal{F}_{Z,W}^\Lambda [b(X), [a(Z), c(W)]] \\ &= (-1)^{aN+N} [[a_\Lambda b]_{\Gamma+\Lambda} c] + (-1)^{(a+N)(b+N)} [b_\Gamma [a_\Lambda c]]. \end{aligned}$$

□

Definition 3.2.3. Let $\mathbb{C}[T, S] := \mathbb{C}[T, S^1, \dots, S^N]$ be the commutative superalgebra freely generated by an even element T and N odd elements S^i . A $N_W = N$ SUSY Lie conformal algebra is a $\mathbb{Z}/2\mathbb{Z}$ -graded $\mathbb{C}[T, S]$ -module \mathcal{R} with a \mathbb{C} -bilinear operation $[\]_\Lambda : \mathcal{R} \otimes_{\mathbb{C}} \mathcal{R} \rightarrow \mathbb{C}[\Lambda] \otimes_{\mathbb{C}} \mathcal{R}$ of parity $N \bmod 2$ satisfying the following three axioms:

(1) Sesquilinearity:

$$\begin{aligned} [Ta_\Lambda b] &= -\lambda[a_\Lambda b] & [a_\Lambda Tb] &= (T + \lambda)[a_\Lambda b] \\ [S^i a_\Lambda b] &= -(-1)^N \chi^i[a_\Lambda b] & [a_\Lambda S^i b] &= (-1)^{a+N} (S^i + \chi^i)[a_\Lambda b] \end{aligned}$$

(2) Skew-symmetry:

$$[b_\Lambda a] = -(-1)^{ab+N} [b_{-\Lambda-\nabla} a],$$

where $\nabla = (T, S^1, \dots, S^N)$, the Λ -bracket in the RHS means compute first the Γ bracket and then let $\Gamma = -\Lambda - \nabla$.

(3) Jacobi identity:

$$(3.2.3.1) \quad [a_\Lambda [b_\Gamma c]] = (-1)^{aN+N} [[a_\Lambda b]_{\Gamma+\Lambda} c] + (-1)^{(a+N)(b+N)} [b_\Gamma [a_\Lambda c]].$$

We will drop the adjective SUSY when no confusion may arise.

Remark 3.2.4. Even though in this case the situation is simple, it is instructive to realize the Λ bracket as a morphism of $\mathbb{C}[\Lambda]$ -modules. Consider the co-commutative Hopf superalgebra $\mathcal{H} = \mathbb{C}[\Lambda]$ with commultiplication $\Delta\lambda = \lambda \otimes 1 + 1 \otimes \lambda$, $\Delta\chi^i = \chi^i \otimes 1 + 1 \otimes \chi^i$. Note that $\mathbb{C}[\nabla] \simeq \mathcal{H}$. Consider \mathcal{H} as a \mathcal{H} -module with the adjoint action (which is trivial in this case, given that \mathcal{H} is super-commutative). Then we may think of $\mathcal{H} \otimes \mathcal{R}$ as an \mathcal{H} module, the action is given by $h \mapsto \Delta h$. Similarly $\mathcal{R} \otimes \mathcal{R}$ is an \mathcal{H} -module. The Λ -bracket is then a \mathcal{H} -module homomorphism of degree $(-1)^N$. Namely, let ϕ denote the morphism $\mathcal{R} \otimes \mathcal{R} \rightarrow \mathcal{H} \otimes \mathcal{R}$ which is given by the Λ -bracket. Then for every $h \in \mathcal{H}$ we have

$$\phi h - (-1)^{hN} h \phi = 0,$$

as elements in $\text{Hom}(\mathcal{R} \otimes \mathcal{R}, \mathcal{H} \otimes \mathcal{R})$. Similarly, the Jacobi identity is an identity in

$$\text{Hom}(\mathcal{R} \otimes \mathcal{R} \otimes \mathcal{R}, \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{R}).$$

We will expand on this in Remark 4.12.

Remark 3.2.5. According to Proposition 3.2.2, given any $N_W = N$ SUSY formal distribution Lie superalgebra $(\mathfrak{g}, \mathcal{R})$, the space \mathcal{R} is a SUSY Lie conformal algebra where $T = \partial_w$ and $S^i = \partial_{\zeta^i}$, and the Λ -bracket is defined by (3.2.1.1).

Definition 3.2.6. A Lie superalgebra of degree $p \in \mathbb{Z}/2\mathbb{Z}$ is a vector superspace \mathfrak{h} with a bilinear operation $\{, \} : \mathfrak{h} \otimes \mathfrak{h} \rightarrow \mathfrak{h}$ of parity p satisfying:

- (1) Skew-symmetry: $\{a, b\} = -(-1)^{ab+p}\{b, a\}$.
- (2) Jacobi identity: $\{a, \{b, c\}\} = (-1)^{ap+p}\{\{a, b\}, c\} + (-1)^{(a+p)(b+p)}\{b, \{a, c\}\}$.

Lemma 3.2.7. Let \mathfrak{h} be a Lie superalgebra of degree $p \in \mathbb{Z}/2\mathbb{Z}$. Define \mathfrak{g} as a vector superspace to be \mathfrak{h} if $p = 0 \pmod{2}$ or \mathfrak{h} with the reversed parity if $p = 1 \pmod{2}$. Define the bilinear operation $[\cdot, \cdot] : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ by:

$$[a, b] = (-1)^{ap+p}\{a, b\},$$

where the right hand side is computed in \mathfrak{h} and then we reverse the parity if $p = \bar{1}$. Then $(\mathfrak{g}, [\cdot, \cdot])$ is a Lie superalgebra which we will denote as $\text{Lie}(\mathfrak{h})$.

Proof. We have:

$$[b, a] = (-1)^{bp+p}\{b, a\} = -(-1)^{bp+ab}\{a, b\} = -(-1)^{(a+p)(b+p)}[a, b],$$

which is skew-symmetry for the Lie algebra provided the parity in \mathfrak{g} is shifted by p . To check Jacobi identity we have:

$$\begin{aligned} [a, [b, c]] &= (-1)^{pb+ap}\{a, \{b, c\}\} = (-1)^{pb+p}\{\{a, b\}, c\} + (-1)^{ab+p}\{b, \{a, c\}\} = \\ &= (-1)^{pb+p+(a+b+p)p+ap}[[a, b], c] + (-1)^{ab+p+ap+bp}[b, [a, c]] = \\ &= [[a, b], c] + (-1)^{(a+p)(b+p)}[b, [a, c]]. \end{aligned}$$

□

Lemma 3.2.8. Let \mathcal{R} be a $N_W = N$ SUSY Lie conformal algebra. Then $\mathcal{R}/\nabla\mathcal{R}$ is naturally a Lie superalgebra of degree $N \pmod{2}$ with bracket

$$\{a + \nabla\mathcal{R}, b + \nabla\mathcal{R}\} = [a_\Lambda b]_{\Lambda=0} + \nabla\mathcal{R}.$$

Proof. The fact that the bilinear map $\{, \}$ is well defined follows from sesquilinearity. Skew-symmetry and the Jacobi identity follow from the corresponding axioms for the SUSY Lie conformal algebra \mathcal{R} . □

Lemma 3.2.9. Let \mathcal{R} be an $N_W = N$ SUSY Lie conformal algebra. Then $\tilde{\mathcal{R}} := \mathcal{R} \otimes \mathbb{C}[W, W^{-1}]$ is an $N_W = N$ SUSY Lie conformal algebra with Λ -bracket:

$$(3.2.9.1) \quad [a \otimes f_\Lambda b \otimes g] = (-1)^{fb}[a_{\Lambda+\partial_w} b] \otimes f(W)g(W')|_{W'=W},$$

and with $\tilde{T} = T \otimes \text{id} + \text{id} \otimes \partial_w$ and $\tilde{S}^i = S^i \otimes \text{id} + \text{id} \otimes \partial_{\zeta^i}$.

Proof. We prove here skew-symmetry, the other axioms are checked in a similar way:

$$\begin{aligned}
[a \otimes f_{\Lambda} b \otimes g] &= (-1)^{fb} [a_{\Lambda+\partial_W} b] \otimes f(W)g(W')|_{W=W'} \\
&= -(-1)^{ab+N+fb} [b_{-\Lambda-\partial_W-\nabla} a] \otimes f(W)g(W')|_{W=W'} \\
&= -(-1)^{(a+f)(b+g)+N+ga} [b_{-\Lambda-\partial_W-\nabla-\partial_{W'}+\partial_{W'}} a] \otimes g(W')f(W)|_{W=W'} \\
&= -(-1)^{(a+f)(b+g)+N} [b \otimes g_{-\Lambda-\tilde{\nabla}} a \otimes f]
\end{aligned}$$

□

3.2.10. For any $N_W = N$ SUSY Lie conformal algebra \mathcal{R} , we put $L(\mathcal{R}) = \tilde{\mathcal{R}}/\tilde{\nabla}\tilde{\mathcal{R}}$ and $\text{Lie}(\mathcal{R}) := \text{Lie}(L(\mathcal{R}))$ (see Lemmas 3.2.7 and 3.2.8). For each $a \in \mathcal{R}$, let $a_{<n|I>} \in L(\mathcal{R})$ be the image of $a \otimes W^{n|I}$. Similarly define $a_{(n|I)} \in \text{Lie}(\mathcal{R})$ as the image of the following element of $L(\mathcal{R})$

$$(-1)^{aI} \sigma(I) a_{<n|I>},$$

and define, for each $a \in \mathcal{R}$, the following $\text{Lie}(\mathcal{R})$ -valued formal distribution

$$(3.2.10.1) \quad a(Z) = \sum_{j \in \mathbb{Z}, J} Z^{-1-j|N \setminus J} a_{(j|J)} \in \text{Lie}(\mathcal{R})[[Z, Z^{-1}]].$$

Using (3.2.9.1) with $f = W^{n|I}$ and $g = W^{k|K}$ and putting $\Lambda = 0$ we compute explicitly the Lie bracket (of parity $N \bmod 2$) in $L(\mathcal{R})$:

$$\begin{aligned}
(3.2.10.2) \quad \{a_{<n|I>}, b_{<k|K>}\} &= \sum_{j \geq 0, J} (-1)^{aJ+b(I-J)} \binom{n}{j} \times \\
&\quad \times \sigma(J, I \setminus J) \sigma(I \setminus J, K) (a_{(j|J)} b)_{<n-j+k|K \cup (I \setminus J)>}.
\end{aligned}$$

It is straightforward to check using Lemma 3.2.7, that the Lie bracket in $\text{Lie}(\mathcal{R})$ is given by:

$$\begin{aligned}
(3.2.10.3) \quad [a_{(n|I)}, b_{(k|K)}] &= (-1)^{(a+N-I)(N-K)} \sum_{(j|J): j \geq 0} (-1)^{(I-J)(N-J)} \binom{n}{j} \times \\
&\quad \times \sigma(I) \sigma(J, I \setminus J) \sigma(I \setminus J, (N \setminus K) \setminus (I \setminus J)) (a_{(j|J)} b)_{(n+k-j|K \cup (I \setminus J))}.
\end{aligned}$$

Proposition 3.2.11. *Let \mathcal{R} be an $N_W = N$ SUSY Lie conformal algebra, a, b two vectors in \mathcal{R} , and $a(Z), b(W)$ the corresponding $\text{Lie}(\mathcal{R})$ -valued formal distributions defined by (3.2.10.1). Then*

$$(3.2.11.1) \quad [a(Z), b(W)] = \sum_{j \geq 0, J} \left(\partial_W^{(j|J)} \delta(Z, W) \right) (a_{(j|J)} b)(W).$$

Proof. First we expand

$$(3.2.11.2) \quad \partial_W^{(j|J)} \delta(Z, W) = \sum_{n \in \mathbb{Z}, I} \binom{n}{j} (-1)^{I-J} \sigma(J) \sigma(N \setminus I, I \setminus J) Z^{-1-n|N \setminus I} W^{n-j|I \setminus J}.$$

Now using (3.2.10.3) we have:

$$(3.2.11.3) \quad [a(Z), b(W)] = \sum_{\substack{n \in \mathbb{Z}, I \\ k \in \mathbb{Z}, K}} \binom{n}{j} (-1)^{(I-J)(N-J)} \sigma(I) \sigma(J, I \setminus J) \times \\ \times \sigma(I \setminus J, (N \setminus K) \setminus (I \setminus J)) Z^{-1-n|N \setminus I} W^{-1-k|N \setminus K} (a_{(j|J)} b)_{(n+k-j|K \cup (I \setminus J))}.$$

On the other hand we have

$$(3.2.11.4) \quad W^{-1-k|N \setminus K} = \sigma(I \setminus J, (N \setminus K) \setminus (I \setminus J)) W^{n-j|I \setminus J} W^{-1-k-n+j|(N \setminus K) \setminus (I \setminus J)},$$

and, due to (3.1.1.1),

$$(3.2.11.5) \quad \sigma(I) \sigma(J, I \setminus J) = (-1)^{(I-J)(N-I)} \sigma(N \setminus I, I \setminus J) \sigma(J).$$

Now substituting (3.2.11.4) in (3.2.11.3) and using (3.2.11.5) we obtain (3.2.11.1). \square

Proposition 3.2.12. *Let \mathcal{R} be an $N_W = N$ SUSY Lie conformal algebra, then the pair $(\text{Lie}(\mathcal{R}), \mathcal{R})$ is an $N_W = N$ SUSY formal distribution Lie superalgebra.*

Proof. The fact that the family of distributions (3.2.10.1) is closed under $(j|J)$ -th products and that they are pairwise local follows from Proposition 3.2.11 since $a_{(j|J)} b = 0$ for $j \gg 0$ in \mathcal{R} . The fact that this family is closed under the derivations $\partial_z, \partial_{\theta^i}$ follows from the following identities which are straightforward to check

$$(3.2.12.1) \quad \begin{aligned} (Ta)_{(j|J)} &= -j a_{(j-1|J)}, \\ (S^i a)_{(j|J)} &= \sigma(e_i, N \setminus J) a_{(j|J \setminus e_i)}. \end{aligned}$$

\square

3.2.13. Note from (3.2.9.1) that $(-\partial_w, -\partial_{\zeta^i})$ are derivations of the $(0|0)$ -th product of $\tilde{\mathcal{R}}$. Since these operators supercommute with (\tilde{T}, \tilde{S}^i) , they induce derivations (T, S^i) of the Lie superalgebra $\text{Lie}(\mathcal{R})$, given by the formulas:

$$(3.2.13.1) \quad \begin{aligned} T(a_{(j|J)}) &= -j a_{(j-1|J)}, \\ S^i(a_{(j|J)}) &= \begin{cases} \sigma(N \setminus J, e_i) a_{(j|J \setminus e_i)} & \text{if } i \in J, \\ 0 & \text{if } i \notin J. \end{cases} \end{aligned}$$

Note that $\text{Lie}(\mathcal{R})$ contains a subalgebra $\text{Lie}(\mathcal{R})_{\leq}$ spanned by vectors $a_{(j|J)}$ with $j \geq 0$. This subalgebra, called the *annihilation subalgebra*, is stable under the action of $\nabla = (T, S^i)$.

Moreover, it is straightforward to check, using (3.2.13.1), that the formal distributions (3.2.10.1) satisfy:

$$(3.2.13.2) \quad Ta(Z) = \partial_z a(Z), \quad S^i a(Z) = \partial_{\theta^i} a(Z)$$

namely, the $N_W = N$ formal distribution Lie superalgebra $(\text{Lie}(\mathcal{R}), \mathcal{R})$ is *regular*.

3.2.14. Recall that we have defined $(j|J)$ -th products of formal distributions for $j \geq 0$ in 3.2.1. In order to define these products for $j < 0$ we let for a formal distribution $a(Z) = \sum Z^{j|J} a_{j|J}$:

$$a_+(Z) = \sum_{(j|J): j \geq 0} Z^{j|J} a_{j|J}, \quad a_-(Z) = \sum_{(j|J): j < 0} Z^{j|J} a_{j|J}$$

It follows easily from the definitions that

$$(3.2.14.1) \quad a_+(W) = \text{res}_Z i_{z,w}(Z - W)^{-1|N} a(Z), \quad a_-(W) = -\text{res}_Z i_{w,z}(Z - W)^{-1|N} a(Z).$$

Indeed, we have

$$i_{z,w}(Z - W)^{-1|N} = \sum_{(m|J):m \geq 0} (-1)^J \sigma(J) W^{m|J} Z^{-1-m|N \setminus J}.$$

Hence:

$$\begin{aligned} \text{res}_Z i_{z,w}(Z - W)^{-1|N} a(Z) &= \text{res}_Z \sum_{\substack{(m|J):m \geq 0 \\ (n|I):n \in \mathbb{Z}}} (-1)^J \sigma(J) W^{m|J} Z^{-1-m|N \setminus J} Z^{n|I} a_{n|I} \\ &= \text{res}_Z \sum_{(m|J):m \geq 0} (-1)^J \sigma(J) \sigma(N \setminus J, J) W^{m|J} Z^{-1|N} a_{m|J} \\ &= \sum_{(m|J):m \geq 0} W^{m|J} a_{m|J} = a_+(W). \end{aligned}$$

The second equation in (3.2.14.1) follows similarly, or by noting that it is a consequence of the first equation in (3.2.14.1), the definition of the δ function (3.1.6.1) and property (5) in 3.1.6. Differentiating (3.2.14.1) we find:

$$(3.2.14.2) \quad \begin{aligned} (-1)^{JN} \partial_W^{(j|J)} a(W)_+ &= \sigma(J) \text{res}_Z i_{z,w}(Z - W)^{-1-j|N \setminus J} a(Z), \\ (-1)^{JN} \partial_W^{(j|J)} a(W)_- &= -\sigma(J) \text{res}_Z i_{w,z}(Z - W)^{-1-j|N \setminus J} a(Z). \end{aligned}$$

These equations (3.2.14.2) are called the super *Cauchy formulae*.

Definition 3.2.15. Let V be a vector superspace. An $\text{End}(V)$ -valued formal distribution $a(Z)$ is called a *field* if for every vector $v \in V$ we have $a(Z)v \in V((Z))$, i.e. there are finitely many negative powers of z in $a(Z)v$. For two such fields we define their *normally ordered product* to be

$$(3.2.15.1) \quad : a(Z)b(Z) : := a_+(Z)b(Z) + (-1)^{ab}b(Z)a_-(Z)$$

3.2.16. The normally ordered product of fields is again a well defined field. Indeed, when applied to any vector $v \in V$ the first summand in (3.2.15.1) clearly has finitely many negative powers of z since $b(Z)v \in V((Z))$ and $a_+(Z)$ has only non-negative powers of z . For the second summand we see that $a_-(Z)v \in V[Z, Z^{-1}]$, namely it is a Laurent polynomial with values in V , therefore $b(Z)a_-(Z)v \in V((Z))$ as we wanted.

Lemma 3.2.17.

$$(3.2.17.1) \quad : a(W)b(W) : := \text{res}_Z \left(i_{z,w}(Z - W)^{-1|N} a(Z)b(W) - (-1)^{ab} i_{w,z}(Z - W)^{-1|N} b(W)a(Z) \right).$$

Proof. This is immediate by (3.2.14.1). □

3.2.18. Given the last lemma and the Cauchy formulae (3.2.14.2) it is natural to define

$$(3.2.18.1) \quad a(W)_{(-1-j|N \setminus J)} b(W) = \sigma(J)(-1)^{JN} : \left(\partial_W^{(j|J)} a(W) \right) b(W) : .$$

Differentiating (3.2.17.1) we find:

$$\begin{aligned} a(W)_{(-1-j|N \setminus J)} b(W) &= \text{res}_Z \left(\left(i_{z,w}(Z - W)^{-1-j|N \setminus J} \right) a(Z) b(W) - \right. \\ &\quad \left. - (-1)^{ab} \left(i_{w,z}(Z - W)^{-1-j|N \setminus J} \right) b(W) a(Z) \right) . \end{aligned}$$

Similarly, from the definition of the $j|J$ -th products for $j \geq 0$ in 3.2.1 we have:

$$\begin{aligned} (3.2.18.2) \quad \text{res}_Z \left(\left(i_{z,w}(Z - W)^{j|J} \right) a(Z) b(W) - (-1)^{ab} \left(i_{w,z}(Z - W)^{j|J} \right) b(W) a(Z) \right) &= \\ = \text{res}_Z (Z - W)^{j|J} (a(Z) b(W) - (-1)^{ab} b(W) a(Z)) &= \\ = \text{res}_Z (Z - W)^{j|J} [a(Z), b(W)] = a(W)_{(j|J)} b(W) . \end{aligned}$$

Therefore we have proved that for every $j \in \mathbb{Z}$ and every tuple J we have:

$$(3.2.18.3) \quad a(W)_{(j|J)} b(W) = \text{res}_Z \left(\left(i_{z,w}(Z - W)^{j|J} \right) a(Z) b(W) - \right. \\ \left. - (-1)^{ab} \left(i_{w,z}(Z - W)^{j|J} \right) b(W) a(Z) \right) .$$

Proposition 3.2.19. *The following identities analogous to sesquilinearity for all pairs $j|J$ are true:*

$$\begin{aligned} (3.2.19.1) \quad & (\partial_w a(W))_{(j|J)} b(W) = -j a(W)_{(-j-1|J)} b(W) \\ & \partial_w (a(W)_{(j|J)} b(W)) = (\partial_w a(W))_{(j|J)} b(W) + a(W)_{(j|J)} \partial_w b(W) \\ & (\partial_{\zeta^i} a(W))_{(j|J)} b(W) = \sigma(J \setminus e_i, e_i) a(W)_{(j|J \setminus e_i)} b(W) \\ & \partial_{\zeta^i} (a(W)_{(j|J)} b(W)) = (-1)^{N-J} \left((\partial_{\zeta^i} a(W))_{(j|J)} b(W) + \right. \\ & \quad \left. + (-1)^a a(W)_{(j|J)} (\partial_{\zeta^i} b(W)) \right) , \end{aligned}$$

where e_i is the tuple consisting of only one element $\{i\}$ and we recall that we are defining $\sigma(e_i, J \setminus e_i)$ to be zero if $i \notin J$.

Proof. The first two equations are standard and their proof is similar to the last two. We will prove the last two equations by using (3.2.18.3). If $i \notin J$ the result is obvious.

$$\begin{aligned} (3.2.19.2) \quad \text{res}_Z i_{z,w}(Z - W)^{j|J} \partial_{\theta^i} a(Z) b(W) &= -(-1)^J \text{res}_Z \left(\partial_{\theta^i} i_{z,w}(Z - W)^{j|J} \right) a(Z) b(W) \\ &= -(-1)^J \sigma(e_i, J \setminus e_i) \text{res}_Z i_{z,w}(Z - W)^{j|J \setminus e_i} a(Z) b(W) . \end{aligned}$$

Similarly we have:

$$\begin{aligned}
 (3.2.19.3) \quad & -(-1)^{(a+1)b} \operatorname{res}_Z i_{w,z}(Z-W)^{j|J} b(W) \partial_{\theta^i} a(Z) = \\
 & = (-1)^{ab+J} \operatorname{res}_Z \left(\partial_{\theta^i} i_{w,z}(Z-W)^{j|J} \right) b(W) a(Z) = \\
 & = (-1)^{ab+J} \sigma(e_i, J \setminus e_i) \operatorname{res}_Z i_{w,z}(Z-W)^{j|J \setminus e_i} b(W) a(Z).
 \end{aligned}$$

Adding (3.2.19.2) and (3.2.19.3) and using (3.1.1.1), we obtain the third equation in (3.2.19.1).

Finally, to prove the last relation in (3.2.19.1) we expand:

$$\begin{aligned}
 (3.2.19.4) \quad & \partial_{\zeta^i} (a(W)_{(j|J)} b(W)) = \partial_{\zeta^i} \operatorname{res}_Z \left(i_{z,w}(Z-W)^{j|J} a(Z) b(W) - \right. \\
 & \quad \left. - (-1)^{ab} i_{w,z}(Z-W)^{j|J} b(W) a(Z) \right) = \\
 & (-1)^N \operatorname{res}_Z \left(-\sigma(e_i, J \setminus e_i) i_{z,w}(Z-W)^{j|J \setminus e_i} a(Z) b(W) + \right. \\
 & \quad + (-1)^{J+a} i_{z,w}(Z-W)^{j|J} a(Z) \partial_{\zeta^i} b(W) - \\
 & \quad - (-1)^{ab} \sigma(e_i, J \setminus e_i) i_{w,z}(Z-W)^{j|J \setminus e_i} b(W) a(Z) - \\
 & \quad \left. - (-1)^{ab+J} i_{w,z}(Z-W)^{j|J} \partial_{\zeta^i} b(W) a(Z) \right) = \\
 & = -(-1)^N \sigma(e_i, J \setminus e_i) a(W)_{(j|J \setminus e_i)} b(W) + (-1)^{N+J+a} a(W)_{(j|J)} \partial_{\zeta^i} b(W) = \\
 & = (-1)^{N-J} \left((\partial_{\zeta^i} a(W))_{(j|J)} b(W) + (-1)^a a(W)_{(j|J)} \partial_{\zeta^i} b(W) \right).
 \end{aligned}$$

□

Proposition 3.2.20. *The following identity holds for any $(j|J)$ and any three fields $a = a(W)$, $b = b(W)$, $c = c(W)$:*

$$\begin{aligned}
 (3.2.20.1) \quad & [a_{\Lambda}(b_{(j|J)} c)] = \\
 & = \sum_{(k|K): k \geq 0} (-1)^{(a+K+N)(J+N)} \sigma(J, K) \Lambda^{(k|K)} [a_{\Lambda} b]_{(j+k|J \cup K)} c + \\
 & \quad + (-1)^{(a+N)(b+N-J)} b_{(j|J)} [a_{\Lambda} c].
 \end{aligned}$$

Proof. The left hand side is

$$\begin{aligned}
 (3.2.20.2) \quad & \operatorname{res}_Z \exp((Z-W)\Lambda) [a(Z), (b(W)_{(j|J)} c(W))] = \\
 & = \operatorname{res}_Z \exp((Z-W)\Lambda) \left([a(Z), \operatorname{res}_X i_{x,w}(X-W)^{j|J} b(X) c(W)] - \right. \\
 & \quad \left. - (-1)^{bc} [a(Z), \operatorname{res}_X i_{w,x}(X-W)^{j|J} c(W) b(X)] \right) = \\
 & (-1)^{a(N-J)} \operatorname{res}_Z \operatorname{res}_X \exp((Z-W)\Lambda) i_{x,w}(X-W)^{j|J} [a(Z), b(X) c(W)] - \\
 & - (-1)^{bc+a(N-J)} \operatorname{res}_Z \operatorname{res}_X \exp((Z-W)\Lambda) i_{w,x}(X-W)^{j|J} [a(Z), c(W) b(X)]
 \end{aligned}$$

Using the identity $[a, bc] = [a, b]c + (-1)^{ab}b[a, c]$ we can write the first term of the RHS of the last equality as:

$$\begin{aligned}
(3.2.20.3) \quad & (-1)^{a(N-J)} \operatorname{res}_Z \operatorname{res}_X \exp((Z - X + X - W)\Lambda) \times \\
& \times i_{x,w}(X - W)^{j|J} [a(Z), b(X)] c(W) + (-1)^{a(N-J+b)} \operatorname{res}_Z \operatorname{res}_X \exp((Z - W)\Lambda) \times \\
& \times i_{x,w}(X - W)^{j|J} b(X) [a(Z), c(W)] = \\
& = (-1)^{a(N-J)+N+JN} \operatorname{res}_X \exp((X - W)\Lambda) i_{x,w}(X - W)^{j|J} [a_\Lambda b](X) c(W) + \\
& + (-1)^{a(N-J+b)+N+JN+bN} \operatorname{res}_X i_{x,w}(X - W)^{j|J} b(X) [a_\Lambda c](W) = \\
& = (-1)^{(a+N)(N-J)} \operatorname{res}_X \sum_{(k|K): k \geq 0} \frac{(-1)^{\frac{K(K+1)}{2}}}{k!} \Lambda^{k|K} \times \\
& \times i_{x,w}(X - W)^{k|K} (X - W)^{j|J} [a_\Lambda b](X) c(W) + \\
& + (-1)^{(a+N)(N-J+b)} \operatorname{res}_X i_{x,w}(X - W)^{j|J} b(X) [a_\Lambda c](W) = \\
& = (-1)^{(a+N)(N-J)} \operatorname{res}_X \sum_{(k|K): k \geq 0} \frac{(-1)^{\frac{K(K+1)}{2}}}{k!} \sigma(K, J) \times \\
& \times \Lambda^{k|K} i_{x,w}(X - W)^{k+j|K \cup J} [a_\Lambda b](X) c(W) + \\
& + (-1)^{(a+N)(N-J+b)} \operatorname{res}_X i_{x,w}(X - W)^{j|J} b(X) [a_\Lambda c](W)
\end{aligned}$$

Similarly the second term in the RHS of the last equality of (3.2.20.2) can be written as:

$$\begin{aligned}
(3.2.20.4) \quad & -(-1)^{bc+a(N-J)} \operatorname{res}_Z \operatorname{res}_X \exp((Z - W)\Lambda) \times \\
& \times i_{w,x}(X - W)^{j|J} [a(Z), c(W)] b(X) - (-1)^{bc+a(N-J+c)} \operatorname{res}_Z \operatorname{res}_X \times \\
& \times \exp((Z - W)\Lambda) i_{w,x}(X - W)^{j|J} c(W) [a(Z), b(X)] = \\
& = -(-1)^{bc+a(N-J)+N+JN} \operatorname{res}_X i_{w,x}(X - W)^{j|J} [a_\Lambda c](W) b(X) - \\
& - (-1)^{bc+(a+N)(N-J+c)} \operatorname{res}_X \exp((X - W)\Lambda) i_{w,x}(X - W)^{j|J} c(W) [a_\Lambda b](X) \\
& = -(-1)^{bc+(a+N)(N-J)} \operatorname{res}_X i_{w,x}(X - W)^{j|J} [a_\Lambda c](W) b(X) - \\
& - (-1)^{bc+(a+N)(N-J+c)} \operatorname{res}_X \sum_{(k|K): k \geq 0} \frac{(-1)^{\frac{K(K+1)}{2}}}{k!} \times \\
& \times \sigma(K, J) \Lambda^{k|K} i_{w,x}(X - W)^{k+j, K \cup J} c(W) [a_\Lambda b](X).
\end{aligned}$$

Now adding (3.2.20.3) and (3.2.20.4) we get (3.2.20.1) (recall that the Λ -bracket has parity $N \bmod 2$). \square

Remark 3.2.21. If we multiply both sides of (3.2.20.1) by

$$\frac{(-1)^{\frac{J(J+1+2a)}{2}}}{j!} \Gamma^{j|J},$$

and sum over all pairs $(j|J)$ with $j \geq 0$ we obtain the Jacobi identity for the Λ -bracket that we have already proved in Proposition 3.2.2. Therefore, the identities (3.2.20.1) for $j \geq 0$ are equivalent to the Jacobi identity (3.2.3.1).

Next we note that if we replace b by $\partial_w b$ in (3.2.20.1) we obtain the same identity with j replaced by $j - 1$ whenever $j \leq -1$. Similarly, replacing b by $\partial_{\zeta^i} b$ we obtain the same identity with J replaced by $J \setminus e_i$. It follows the identity (3.2.20.1) is equivalent to the Jacobi identity (3.2.3.1) and (3.2.20.1) with $(j|J) = (-1|N)$. In this case the formula (3.2.20.1) looks as follows:

$$[a_\Lambda : bc :] = \sum_{k \geq 0} \frac{\lambda^k}{k!} [a_\Lambda b]_{(k-1|N)} c + (-1)^{(a+N)b} : b[a_\Lambda c] :$$

Rewriting the sum as the sum of the $k = 0$ term and the rest, this becomes:

$$(3.2.21.1) \quad [a_\Lambda : bc :] = : [a_\Lambda b] c : + (-1)^{(a+N)b} : b[a_\Lambda c] : + \int_0^\Lambda [[a_\Lambda b]_\Gamma c] d\Gamma.$$

Here the integral \int_0^Λ is computed by taking the indefinite integral in the even variable γ of ∂_γ^N of the integrand, and then taking the difference of the values at the limits. This is the super analogue of the *non-commutative Wick formula* [Kac96]. Thus, the identity (3.2.20.1) is equivalent to the Jacobi identity plus this non-commutative Wick formula.

The following lemma is proved as in the ordinary vertex algebra case [Kac96, Lem. 3.2].

Lemma 3.2.22 (Dong's Lemma). *Given three pairwise local formal distributions a, b, c , the pair $(a, b_{(j|J)} c)$ is local for any $(j|J)$.*

3.3. Identities and existence theorem. In this section we define $N_W = N$ SUSY vertex algebras, derive their identities, and prove an existence theorem as in the non-super case [Kac96, Thm. 4.5].

Definition 3.3.1. An $N_W = N$ SUSY vertex algebra consists of a vector superspace V , an even vector $|0\rangle \in V$, N odd operators S^i (the odd translation operators), an even operator T (the even translation operator), and a parity preserving linear map Y from V to the space of $\text{End}(V)$ -valued superfields $a \mapsto Y(a, Z)$. The following axioms must be satisfied:

- Vacuum axioms:

$$Y(a, Z)|0\rangle = a + O(Z), \quad T|0\rangle = S^i|0\rangle = 0, \quad i = 1, \dots, N.$$

- Translation invariance

$$(3.3.1.1) \quad [S^i, Y(a, Z)] = \partial_{\theta^i} Y(a, Z), \quad [T, Y(a, Z)] = \partial_Z Y(a, Z).$$

- Locality

$$(z - w)^n [Y(a, Z), Y(b, W)] = 0 \quad \text{for some } n \in \mathbb{Z}_+.$$

As before, $O(Z)$ is an element of $V[[Z]]$ which vanishes at $Z = 0$.

Morphisms between $N_W = N$ SUSY vertex algebras are linear maps $f : V_1 \rightarrow V_2$ such that:

$$f \circ T_1 = T_2 \circ f, \quad f(Y_1(a, Z)b) = Y(f(a), Z)f(b), \quad \forall a, b \in V_1.$$

3.3.2. Given a $N_W = n$ SUSY vertex algebra V , we can define the $(j|J)$ product of two vectors of V as follows. Expand the field $Y(a, Z)$ for $a \in V$:

$$(3.3.2.1) \quad Y(a, Z) = \sum_{(j|J): j \in \mathbb{Z}} Z^{-1-j|N \setminus J} a_{(j|J)},$$

and define the $j|J$ -product of two vectors in V as:

$$(3.3.2.2) \quad a_{(j|J)} b := a_{(j|J)}(b).$$

This is a \mathbb{C} -bilinear product on V of parity $N - J \bmod 2$. We can rewrite the axioms of the vertex algebra in terms of these products. For example, the vacuum axioms are equivalent to:

$$(3.3.2.3) \quad a_{(-1|N)}|0\rangle = a, \quad a_{(j|J)}|0\rangle = 0 \quad \text{if } j \geq 0,$$

$$(3.3.2.4) \quad T|0\rangle = S^i|0\rangle = 0.$$

and translation invariance is equivalent to:

$$(3.3.2.5) \quad \begin{aligned} [T, a_{(j|J)}] &= -j a_{(j-1|J)}, \\ [S^i, a_{(j|J)}] &= \begin{cases} \sigma(N \setminus J, e_i) a_{(j|J \setminus e_i)} & \text{if } i \in J, \\ 0 & \text{if } i \notin J. \end{cases} \end{aligned}$$

Of course the fact that $Y(a, Z)$ is a field is equivalent to $a_{(j|J)} b = 0$ for $j \gg 0$, given $a, b \in V$.

Theorem 3.3.3. *Let \mathcal{U} be a vector superspace and V a space of pairwise local $\text{End}(\mathcal{U})$ -valued fields such that V contains the constant field Id , it is invariant under the derivations $\partial_z, \partial_{\theta^i}$ and closed under all $(j|J)$ -th products. Then V is a $N_W = N$ SUSY vertex algebra with vacuum vector Id , translation operators $Ta(Z) = \partial_z a(Z)$ and $S^i a(Z) = \partial_{\theta^i} a(Z)$, and the $(j|J)$ products are given by the RHS of (3.2.18.3) multiplied by $\sigma(J)^4$.*

Proof. To check the vacuum axioms we have:

$$\begin{aligned} a(Z)_{(j|J)} 1 &= \sigma(J) \text{res}_Z (Z - W)^{j|J} [a(Z), 1] = 0 \quad \text{if } j \geq 0, \\ a(Z)_{(-1|N)} 1 &=: a(Z) 1 := a(Z), \quad \partial_z 1 = \partial_{\theta^i} 1 = 0. \end{aligned}$$

To check translation invariance we have:

$$\partial_z (a(Z)_{(j|J)} b(Z)) - a(Z)_{(j|J)} \partial_z b(Z) = (\partial_z a(Z))_{(j|J)} b(Z),$$

but this is $-j a(Z)_{(j-1|J)} b(Z)$, according to (3.2.19.1). Therefore we see that the first equation in (3.3.2.5) holds. For the odd translation operators we write (note that the parity of $a_{(j|J)}$ is $a + N - J$ since Y is parity preserving and our choice of decomposing the field in (3.3.2.1)):

$$\begin{aligned} \sigma(J) \left(\partial_{\theta^i} (a(Z)_{(j|J)} b(Z)) - (-1)^{a+N-J} a(Z)_{(j|J)} \partial_{\theta^i} b(Z) \right) &= \\ &= (-1)^{N-J} \sigma(J) (\partial_{\theta^i} a(Z))_{(j|J)} b(Z), \end{aligned}$$

and again by (3.2.19.1) we see that this is

$$-(-1)^N \sigma(J) \sigma(e_i, J \setminus e_i) a(Z)_{(j|J \setminus e_i)} b(Z) = \sigma(N \setminus J, e_i) \sigma(J \setminus e_i) a(Z)_{(j|J \setminus e_i)} b(Z),$$

⁴This normalization becomes necessary because of our choice in (3.2.18.1), see also Theorem 3.3.9

proving the second identity in (3.3.2.5). In order to check locality, we expand (3.3.3.1)

$$\begin{aligned}
Y(a(W), X)b(W) &= \sum_{(j|J): j \in \mathbb{Z}} \sigma(J) X^{-1-j|N \setminus J} a(W)_{(j|J)} b(W) \\
&= \text{res}_Z \sum_{(j|J): j \in \mathbb{Z}} (-1)^{(N-J)N} \sigma(J) X^{-1-j|N \setminus J} \times \\
&\quad \times \left(i_{z,w}(Z-W)^{j|J} a(Z) b(W) - (-1)^{ab} i_{w,z}(Z-W)^{j|J} b(W) a(Z) \right) \\
&= \text{res}_Z \sum_{(j|J): j \in \mathbb{Z}} (-1)^{N-J} \sigma(J) \left(i_{z,w}(Z-W)^{j|J} X^{-1-j|N \setminus J} \times \right. \\
&\quad \left. \times a(Z) b(W) - (-1)^{ab} i_{w,z}(Z-W)^{j|J} X^{-1-j|N \setminus J} b(W) a(Z) \right).
\end{aligned}$$

We note that

$$(3.3.3.2) \quad i_{z,w} \sum_{(j|J): j \in \mathbb{Z}} (-1)^{(N-J)} \sigma(J) (Z-W)^{j|J} X^{-1-j|N \setminus J} = i_{z,w} \delta(Z-W, X).$$

Therefore the RHS of (3.3.3.1) reads:

$$\text{res}_Z \left(i_{z,w} \delta(Z-W, X) a(Z) b(W) - (-1)^{ab} i_{w,z} \delta(Z-W, X) b(W) a(Z) \right).$$

With this last equation we can compute then the commutator $[Y(a(W)), Y(b(W))]c(W)$. Indeed, the product $Y(a(W), X)Y(b(W), Y)c(W)$ is given by:

$$\begin{aligned}
(3.3.3.3) \quad \text{res}_Z \text{res}_U &\left(i_{u,w} i_{z,w} \delta(U-W, X) \delta(Z-W, Y) a(U) b(Z) c(W) - \right. \\
&\quad - (-1)^{bc} i_{u,w} i_{w,z} \delta(U-W, X) \delta(Z-W, Y) a(U) c(W) b(Z) - \\
&\quad - (-1)^{a(b+c)} i_{w,u} i_{z,w} \delta(U-W, X) \delta(Z-W, Y) b(Z) c(W) a(U) + \\
&\quad \left. + (-1)^{a(b+c)+bc} i_{w,u} i_{w,z} \delta(U-W, X) \delta(Z-W, Y) c(W) b(Z) a(U) \right),
\end{aligned}$$

and we get a similar expression for the product $Y(b(W), Y)Y(a(W), X)c(W)$. Subtracting we obtain:

$$\begin{aligned}
(3.3.3.4) \quad [Y(a(W), X), Y(b(W), Y)]c(W) &= \\
&\text{res}_Z \text{res}_U \left(i_{u,w} i_{z,w} \delta(U-W, X) \delta(Z-W, Y) [a(U), b(Z)] c(W) - \right. \\
&\quad \left. - (-1)^{(a+b)c} i_{w,u} i_{w,z} \delta(U-W, X) \delta(Z-W, Y) c(W) [a(U), b(Z)] \right).
\end{aligned}$$

Let $n \in \mathbb{Z}_+$ be such tht $(u-z)^n [a(U), b(Z)] = 0$. Multiplying (3.3.3.4) by $(x-y)^n$ we obtain that the RHS vanishes. Indeed, using

$$(x-y) = (z-u) - ((z-w) - x) + ((u-w) - y),$$

we see that all terms in the expansion of $(x-y)^n$ vanish when multiplied by δ functions, with the exception of $(z-u)^n$. But this term vanishes when multiplied by the factors $[a(U), b(Z)]$ in (3.3.3.4). Therefore we have proved locality and the theorem. \square

Corollary 3.3.4. *Any identity on elements of an $N_W = N$ SUSY vertex algebra, holds for any collection of pairwise local fields.*

Lemma 3.3.5. *Let V be a vector superspace and let $|0\rangle$ be an even vector of V . Let $a(Z), b(Z)$ be two $\text{End}(V)$ -valued fields such that $a(Z)|0\rangle \in V[[Z]]$ and $b(Z)|0\rangle \in V[[Z]]$. Then for all $(j|J)$, $a(W)_{(j|J)}b(W)|0\rangle \in V[[W]]$ and the constant term is*

$$(3.3.5.1) \quad \sigma(J)a_{(j|J)}b_{(-1|N)}|0\rangle.$$

Proof. Applying both sides of (3.2.18.3) to the vacuum, we see that the second term on the RHS of (3.2.18.3) vanishes since it contains only positive powers of z . The first term in the RHS contains only positive powers of w since $i_{z,w}(Z-W)^{j|J}$ does and $b(W)|0\rangle \in \mathbb{C}[[W]]$. Letting $W=0$ we get

$$(3.3.5.2) \quad a(W)_{(j|J)}b(W)|0\rangle|_{W=0} = \text{res}_Z Z^{j|J}a(Z)(b_{(-1|N)}|0\rangle).$$

It follows from (3.1.2.2) that the RHS of (3.3.5.2) is (3.3.5.1). \square

The following lemma is straightforward

Lemma 3.3.6. *Let A and B_1, \dots, B_N be linear operators on a vector superspace \mathcal{U} . Suppose that A is even and B_i are odd and they pairwise (super) commute, i.e. $AB_i = B_iA$, $B_iB_j = -B_jB_i$. Then there exists a unique solution $f(Z) \in \mathcal{U}[[Z]]$ to the system of differential equations:*

$$(3.3.6.1) \quad \partial_z f(Z) = Af(Z), \quad \partial_{\theta^i} f(Z) = B_i f(Z) \quad (i = 1, \dots, N),$$

for any initial condition $f(0) = f_0$.

Proof. Using (3.3.6.1), the coefficients of $f(Z)$ can be computed by induction, given f_0 . \square

Proposition 3.3.7. *Let V be a $N_W = N$ SUSY vertex algebra. Then for every $a, b \in V$:*

- (a) $Y(a, Z)|0\rangle = \exp(Z\nabla)a$.
- (b) $\exp(Z\nabla)Y(a, W)\exp(-Z\nabla) = i_{w,z}Y(a, Z+W)$.
- (c) $Y(a, Z)_{(j|J)}Y(b, Z)|0\rangle = \sigma(J)Y(a_{(j|J)}b, Z)|0\rangle$,

where $\nabla = (T, S^1, \dots, S^N)$ and $Z\nabla = zT + \sum_i \theta^i S^i$.

Proof. We note that both sides in (a) and (c) are elements of $V[[Z]]$ whereas both sides of (b) are elements of $\text{End}(V)[[W, W^{-1}]][[Z]]$. Note that by evaluating at $Z=0$ we get equalities in all three cases, the only non-trivial case is (c), but it follows from Lemma 3.3.5. Let us denote the right hand side in each case by $X(Z)$. It is easy to show that it satisfies the following systems of equations respectively:

- (1) $\partial_z X(Z) = TX(Z)$, and $\partial_{\theta^i} X(Z) = S^i X(Z)$.
- (2) $\partial_z X(Z) = [T, X(Z)]$ and $\partial_{\theta^i} X(Z) = [S^i, X(Z)]$ by the translation axioms.
- (3) $\partial_z X(Z) = TX(Z)$ and $\partial_{\theta^i} X(Z) = S^i X(Z)$ by the translation axioms (recall that $T|0\rangle = S^i|0\rangle = 0$).

In order to apply Lemma 3.3.6, we have to show that the left hand side of (a), (b) and (c) satisfies the same differential equations (1), (2) and (3) respectively;

- (1) It is immediate by the translation invariance and the second of the vacuum axioms.

(2) Denoting $Y(Z) = e^{Z\nabla}Y(a, W)e^{-Z\nabla}$, we have:

$$\partial_z Y(Z) = TY(Z) - Y(Z)T = [T, Y(Z)],$$

and similarly:

$$\begin{aligned}\partial_{\theta^i} Y(Z) &= S^i Y(Z) + (-1)^a e^{Z\nabla} Y(a, W) (-S^i) e^{-Z\nabla} \\ &= S^i Y(Z) - (-1)^a Y(Z) S^i = [S^i, Y(Z)].\end{aligned}$$

(3) Denote $Y(Z) = Y(a, Z)_{(j|J)} Y(b, Z)|0\rangle$ and recall that from Proposition 3.2.19, ∂_z and ∂_{θ^i} are derivations of the $(j|J)$ products. To simplify notation, we will denote $a(Z) = Y(a, Z)$ and $b(Z) = Y(b, Z)$. We have:

$$\begin{aligned}S^i Y(W) &= S^i \text{res}_Z \left(i_{z,w}(Z - W)^{j|J} a(Z) b(W) |0\rangle - \right. \\ &\quad \left. - (-1)^{ab} i_{w,z}(Z - W)^{j|J} b(W) a(Z) |0\rangle \right) = \\ &= (-1)^{N+J} \text{res}_Z \left(i_{z,w}(Z - W)^{j|J} [S^i, a(Z)] b(W) |0\rangle \right. \\ &\quad \left. + (-1)^a i_{z,w}(Z - W)^{j|J} a(Z) [S^i, b(W)] |0\rangle - (-1)^{ab} i_{w,z}(Z - W)^{j|J} [S^i, b(W)] a(Z) |0\rangle - \right. \\ &\quad \left. - (-1)^{ab+b} i_{w,z}(Z - W)^{j|J} b(W) [S^i, a(Z)] |0\rangle \right),\end{aligned}$$

and, using $S^i |0\rangle = 0$,

$$\begin{aligned}&= (-1)^{N+J} \text{res}_Z \left(i_{z,w}(Z - W)^{j|J} (\partial_{\theta^i} a(Z)) b(W) |0\rangle + \right. \\ &\quad \left. + (-1)^a i_{z,w}(Z - W)^{j|J} a(Z) (\partial_{\zeta^i} b(W)) |0\rangle - \right. \\ &\quad \left. - (-1)^{ab} i_{w,z}(Z - W)^{j|J} (\partial_{\zeta^i} b(W)) a(Z) |0\rangle - (-1)^{ab+b} i_{w,z}(Z - W)^{j|J} b(W) (\partial_{\theta^i} a(Z)) |0\rangle \right) \\ &= (-1)^{N+J} \left((\partial_{\zeta^i} a(W))_{(j|J)} b(W) + (-1)^a a(W)_{(j|J)} (\partial_{\zeta^i} b(W)) \right) |0\rangle = \\ &= \partial_{\zeta^i} (a(W)_{(j|J)} b(W) |0\rangle).\end{aligned}$$

The proof for T is similar. \square

Proposition 3.3.8 (Uniqueness). *Let V be a $N_W = N$ SUSY vertex algebra and let $a(Z)$ be an $\text{End}(V)$ -valued field such that the pair $(a(Z), Y(b, Z))$ is local for every $b \in V$, and $a(Z)|0\rangle = 0$, then $a(Z) = 0$.*

Proof. By locality there exists $n \in \mathbb{Z}_+$ such that

$$(z - w)^n a(Z) Y(b, W) |0\rangle = (-1)^{ab} (z - w)^n Y(b, W) a(Z) |0\rangle = 0.$$

By Proposition 3.3.7 (1), the left hand side is $(z - w)^n a(Z) e^{W\nabla} b$. Letting $W = 0$, we get $z^n a(Z) b = 0$, and this holds for all b , therefore $a(Z) = 0$. \square

As a simple corollary of the previous proposition and Proposition 3.3.7 we obtain the following

Theorem 3.3.9. *In an $N_W = N$ SUSY vertex algebra the following identities hold*

- (1) $Y(a_{(j|J)} b, Z) = \sigma(J) Y(a, Z)_{(j|J)} Y(b, Z)$ ($(j|J)$ -th product identity).
- (2) $Y(a_{(-1|N)} b, Z) =: Y(a, Z) Y(b, Z) :.$
- (3) $Y(Ta, Z) = \partial_z Y(a, Z)$.
- (4) $Y(S^i a, Z) = \partial_{\theta^i} Y(a, Z)$.

(5) We have the following OPE formula (the sums are finite):
 (3.3.9.1)

$$\begin{aligned} [Y(a, Z), Y(b, W)] &= \sum_{(j|J): j \geq 0} \sigma(J) (\partial_W^{(j|J)} \delta(Z, W)) Y(a_{(j|J)} b, W) \\ &= \sum_{(j|J): j \geq 0} (i_{z,w} - i_{w,z}) (Z - W)^{-1-j|N \setminus J} Y(a_{(j|J)} b, W). \end{aligned}$$

Proof. (1) is the combined statement of Dong's Lemma 3.2.22, and Propositions 3.3.8 and 3.3.7 (c). (2) follows from (1) by letting $j|J = -1|N$. To prove (3) we write, using (3.3.2.5), the $(-2|N)$ -product identity, (3.2.18.1) and the vacuum axiom:

$$Y(Ta, Z) = Y(a_{(-2,N)} |0\rangle, Z) = Y(a, Z)_{(-2|N)} \text{Id} =: \partial_z Y(a, Z) \text{Id} := \partial_z Y(a, Z).$$

(4) follows similarly:

$$\begin{aligned} Y(S^i a, Z) &= Y(a_{(-1, N \setminus e_i)} |0\rangle, Z) = \\ &= -\sigma(N \setminus e_i, e_i) \sigma(e_i, N \setminus e_i) (-1)^N : \partial_{\theta^i} Y(a, Z) \text{Id} := \partial_{\theta^i} Y(a, Z). \end{aligned}$$

Finally (5) follows from (1) and the decomposition Lemma 3.1.7 \square

Corollary 3.3.10. *Let $e_i = \{i\}$. One has (cf. (3.2.19.1)):*

$$\begin{aligned} (Ta)_{(j|J)} &= -j a_{(j-1|J)}, \quad (S^i a)_{(j|J)} = \sigma(e_i, N \setminus J) a_{(j|J \setminus e_i)}, \\ T(a_{(j|J)} b) &= (Ta)_{(j|J)} b + a_{(j|J)} T(b), \\ S^i(a_{(j|J)} b) &= (-1)^{N-J} \left((S^i a)_{(j|J)} b + (-1)^a a_{(j|J)} S^i b \right). \end{aligned}$$

Lemma 3.3.11.

$$i_{x,z} \delta(X - Z, W) = i_{w,z} \delta(X, W + Z).$$

Proof. For simplicity let us assume $N = 0$, the general result follows easily. Denote:

$$\begin{aligned} \psi &= i_{x,z} i_{x-z,w} (x - w - z)^{-1} \in \mathbb{C}[[x, x^{-1}, z, z^{-1}, w, w^{-1}]], \\ \varphi &= i_{w,z} i_{x,w+z} (x - w - z)^{-1} \in \mathbb{C}[[x, x^{-1}, z, z^{-1}, w, w^{-1}]] \end{aligned}$$

It is straightforward to check that both ψ and φ are elements of $K[[z, w]]$ where $K = \mathbb{C}((x))$. On the other hand, since both compositions $i_{x,z} i_{x-z,w}$ and $i_{w,z} i_{x,w+z}$ commute with multiplication by x, z and w , we have $(x - w - z)(\psi - \varphi) = 0$, hence $\psi = \varphi$, since $K[[z, w]]$ has no zero divisors. Similarly, we have:

$$(i_{x,z} i_{w,x-z} - i_{w,z} i_{w+z,x})(x - w - z)^{-1} = 0,$$

and the lemma follows. \square

3.3.12. Taking the generating series in Theorem 3.3.9(1) we obtain for the left hand side:

$$\sum_{(j|J): j \in \mathbb{Z}} W^{-1-j|N \setminus J} Y(a_{(j|J)} b, Z) = Y(Y(a, W) b, Z).$$

On the right hand side we obtain

$$\begin{aligned}
& \sum_{(j|J):j \in \mathbb{Z}} W^{-1-j|N \setminus J} \sigma(J) \operatorname{res}_X \left(i_{x,z}(X-Z)^{j|J} Y(a, X) Y(b, Z) - \right. \\
& \quad \left. - (-1)^{ab} i_{z,x}(X-Z)^{j|J} b(Z) a(X) \right) = \\
& = \operatorname{res}_X \sum_{(j|J):j \in \mathbb{Z}} (-1)^{N \setminus J} \sigma(J) \left(i_{x,z}(X-Z)^{j|J} W^{-1-j|N \setminus J} Y(a, X) Y(b, Z) - \right. \\
& \quad \left. - (-1)^{ab} i_{z,x}(X-Z)^{j|J} W^{-1-j|N \setminus J} Y(b, Z) Y(a, X) \right).
\end{aligned}$$

But, according to (3.3.3.2), this is

(3.3.12.1)

$$\operatorname{res}_X \left(i_{x,z} \delta(X-Z, W) Y(a, X) Y(b, Z) - (-1)^{ab} i_{z,x} \delta(X-Z, W) Y(b, Z) Y(a, X) \right).$$

Using Lemma 3.3.11, the first term gives

$$(3.3.12.2) \quad i_{w,z} Y(a, W+Z) Y(b, Z).$$

In order to compute the second term we expand in Taylor series (cf. 3.1.3.2)

$$i_{z,x} \delta(X-Z, W) = \sum_{(k|K):k \geq 0} (-1)^K X^{k|K} \partial_{-Z}^{(k|K)} \delta(-Z, W).$$

Hence the second term in (3.3.12.1) reads:

$$\begin{aligned}
& -(-1)^{ab} \operatorname{res}_X \sum_{(k|K):k \geq 0} (-1)^K X^{k|K} \partial_Z^{(k|K)} \delta(-Z, W) Y(b, Z) X^{-1-n|N \setminus I} a_{(n|I)} = \\
& = -\operatorname{res}_X \sum_{(k|K):k \geq 0} (-1)^{ab+(N-I)(b+N-K)+K} \sigma(K, N \setminus I) \times \\
& \quad \times X^{k-1-n|K \cup (N \setminus I)} \partial_{-Z}^{(k|K)} \delta(-Z, W) Y(b, Z) a_{(n|I)} = \\
& = - \sum_{(k|K):k \geq 0} (-1)^{(a+N-K)b+N} \sigma(K) \partial_{-Z}^{(k|K)} \delta(-Z, W) Y(b, Z) a_{(k|K)}.
\end{aligned}$$

Adding this to (3.3.12.2) and changing Z by $-Z$ we obtain the important formula

$$\begin{aligned}
(3.3.12.3) \quad & Y \left(Y(a, W) b, -Z \right) = i_{w,z} Y(a, W-Z) Y(b, -Z) - \\
& - \sum_{(k|K):k \geq 0} (-1)^{(a+N-K)b+N} \sigma(K) \partial_Z^{(k|K)} \delta(Z, W) Y(b, -Z) a_{(k|K)}.
\end{aligned}$$

Note now that by acting on any vector $c \in V$ and multiplying this last equation by a sufficiently high power of $(z-w)$ the second term vanishes, therefore we obtain *associativity* for the vertex operators, namely:

$$(3.3.12.4) \quad (z-w)^n Y \left(Y(a, W) b, -Z \right) c = (z-w)^n Y(a, W-Z) Y(b, -Z) c, \quad n \gg 0.$$

As in [FBZ01, 3.2.3] we obtain an equivalent formulation which is called the *Cousin* property. Recall the embedding:

$$i_{z,w} : \mathbb{C}((Z, W)) \hookrightarrow \mathbb{C}((Z))((W))$$

Given $f \in \mathbb{C}((Z, W))$, $i_{z,w} f$ is called the *expansion of f in the domain $|z| > |w|$* .

Corollary 3.3.13 (Cousin property). *For any $N_W = n$ SUSY vertex algebra V and vectors $a, b, c \in V$, the three expressions:*

$$\begin{aligned} Y(a, Z)Y(b, W)c &\in V((Z))((W)) \\ (-1)^{ab}Y(b, W)Y(a, Z)c &\in V((W))((Z)) \\ Y(Y(a, Z - W)b, W)c &\in V((W))((Z - W)) \end{aligned}$$

are the expansions, in the domains $|z| > |w|$, $|w| > |z|$ and $|w| > |w - z|$ respectively, of the same element of $V[[Z, W]][z^{-1}, w^{-1}, (z - w)^{-1}]$.

Proof. By the locality axiom, there exists $n \in \mathbb{Z}_+$ such that:

$$(z - w)^n Y(a, Z)Y(b, W)c = (-1)^{ab}(z - w)^n Y(b, W)Y(a, Z)c.$$

Since the LHS is an element of $V((Z))((W))$ and the RHS is an element of $V((W))((Z))$, it follows that they are both equal to some $\varphi \in V[[Z, W]][z^{-1}, w^{-1}]$ (cf. (3.1.2.3)). Since $i_{z,w}$ and $i_{w,z}$ are algebra morphisms, we get

$$Y(a, Z)Y(b, W)c = i_{z,w} \frac{\varphi}{(z - w)^n}, \quad (-1)^{ab}Y(b, W)Y(a, Z)c = i_{w,z} \frac{\varphi}{(z - w)^n}.$$

The rest of the corollary is proved in a similar way, using (3.3.12.4). \square

Theorem 3.3.14 (Skew-symmetry). *In an $N_W = N$ SUSY vertex algebra the following identity, called skew-symmetry, holds*

$$(3.3.14.1) \quad Y(a, Z)b = (-1)^{ab}e^{Z\nabla}Y(b, -Z)a$$

Proof. By the locality axiom we have for $n \gg 0$

$$(z - w)^n Y(a, Z)Y(b, W)|0\rangle = (z - w)^n (-1)^{ab}Y(b, W)Y(a, Z)|0\rangle$$

Now by (1) in Proposition 3.3.7 we can write this as:

$$\begin{aligned} (3.3.14.2) \quad (z - w)^n Y(a, Z)e^{W\nabla}b &= (z - w)(-1)^{ab}Y(b, W)e^{Z\nabla}a \\ &= (z - w)^n (-1)^{ab}e^{Z\nabla}e^{-Z\nabla}Y(b, W)e^{Z\nabla}a = (z - w)^n (-1)^{ab}e^{Z\nabla}i_{w,z}Y(b, W - Z)a, \end{aligned}$$

where in the last line we used (2) of Proposition 3.3.7. Now both sides in (3.3.14.2) are formal power series in W . Indeed, since $b_{(j|J)}a = 0$ for $j \gg 0$ we see that by making n large enough we may assume that there are no negative powers of w in the RHS. We can then let $W = 0$ in (3.3.14.2) and multiply by z^{-n} to obtain (3.3.14.1). \square

3.3.15. Expanding both sides in (3.3.14.1) we have:

$$\begin{aligned} \sum_{(j|J):j \in \mathbb{Z}} Z^{-1-j|N \setminus J} a_{(j|J)}b &= (-1)^{ab} \left(\sum_{(j|J):j \geq 0} \nabla^{(j|J)} Z^{j|J} \right) \times \\ &\times \left(\sum_{(k|K):k \in \mathbb{Z}} (-Z)^{-1-k|N \setminus K} b_{(k|K)}a \right) = (-1)^{ab} \sum_{\substack{(j|J):j \geq 0 \\ (k|K):k \in \mathbb{Z}}} (-1)^{1+k+N-K} \nabla^{(j|J)} \times \\ &\times \sigma(J, N \setminus K) Z^{j-1-k|J \cup (N \setminus K)} b_{(k|K)}a \end{aligned}$$

Taking the coefficient of $Z^{-1-n|N \setminus I}$ on both sides we get:

$$(3.3.15.1) \quad a_{(n|I)}b = (-1)^{ab} \sum_{j \geq 0, J \cap I = \emptyset} (-1)^{1-n+N+J-I} \times \\ \times (-\nabla)^{(j|J)} \sigma(N \setminus (I \cup J), J) b_{(n+j|I \cup J)} a.$$

In particular, when $(n|I) = (-1|N)$ in (3.3.15.1), we get:

$$:ab: - (-1)^{ab} :ba: = (-1)^{ab} \sum_{j \geq 1} \frac{(-T)^j}{j!} (b_{(-1+j|N)} a),$$

or, equivalently, after exchanging a and b :

$$(3.3.15.2) \quad :ab: - (-1)^{ab} :ba: = \int_{-\nabla}^0 [a_\Lambda b] d\Lambda.$$

Identity (3.3.15.2) is called the *quasi-commutativity* of the normally ordered product.

3.3.16. Define the following *formal Fourier transform* by

$$F_Z^\Lambda a(Z) = \text{res}_Z e^{Z\Lambda} a(Z).$$

It is a linear map from the space of \mathcal{U} -valued formal distributions in Z to $\mathcal{U}[[\Lambda]]$. It has the following properties which are immediate to check:

$$(3.3.16.1) \quad F_Z^\Lambda \partial_z a(Z) = -\lambda F_Z^\Lambda a(Z),$$

$$(3.3.16.2) \quad F_Z^\Lambda \partial_{\theta^i} a(Z) = -(-1)^N \chi^i F_Z^\Lambda a(Z),$$

$$(3.3.16.3) \quad F_Z^\Lambda (e^{Z\nabla} a(Z)) = F_Z^{\Lambda+\nabla} a(Z) \text{ if } a(Z) \in \mathcal{U}((Z)),$$

$$(3.3.16.4) \quad F_Z^\Lambda a(-Z) = -F_Z^{-\Lambda} a(Z),$$

$$(3.3.16.5) \quad F_Z^\Lambda \left(\partial_W^{(j|J)} \delta(Z, W) \right) = (-1)^{JN} e^{W\Lambda} \Lambda^{(j|J)}.$$

Theorem 3.3.17. *Let V be a $N_W = N$ SUSY vertex algebra. Then V is a $N_W = N$ SUSY Lie conformal algebra with Λ -bracket:*

$$(3.3.17.1) \quad [a_\Lambda b] = F_Z^\Lambda Y(a, Z)b = \sum_{(j|J): j \geq 0} (-1)^{JN} \sigma(J) \Lambda^{(j|J)} (a_{(j|J)} b).$$

Proof. The sesquilinearity relations follow from Corollary 3.3.10 for $j \geq 0$. Applying F_Z^Λ to both sides of (3.3.14.1) and using (3.3.16.3) and (3.3.16.4) we get the skew-symmetry relation. In order to prove the Jacobi identity, apply F_Z^Λ to the OPE formula (3.3.9.1) applied to c , and use (3.3.16.5) to obtain (cf. (3.2.20.1)):

$$[a_\Lambda Y(b, W)c] = (-1)^{ab+bN} Y(b, W)[a_\Lambda c] + e^{W\Lambda} Y([a_\Lambda b], W)c.$$

Applying F_W^Γ to both sides of this formula we get the Jacobi identity. \square

Theorem 3.3.18. *Let V be a $N_W = N$ SUSY vertex algebra. The following identity called “quasi-associativity” of the normally ordered product holds for every $a, b, c \in V$:*

$$::ab:c:-:a:bc:: = \sum_{j \geq 0} a_{(-2-j|N)} (b_{(j|N)} c) + (-1)^{ab} \sum_{j \geq 0} b_{(-2-j|N)} (a_{(j|N)} c).$$

Equivalently

$$:: ab : c : - : a : bc ::= \left(\int_0^\nabla d\Lambda a \right) [b_\Lambda c] + (-1)^{ab} \left(\int_0^\nabla d\Lambda b \right) [a_\Lambda c],$$

where the integral is computed as follows: expand the Λ -bracket, put the powers of Λ on the left, under the sign of integral, then take the definite integral by the usual rules inside the parenthesis.

Proof. Applying both sides of Theorem 3.3.9 (2) to c and taking the constant coefficient, the LHS is $:: ab : c :$. By (3.2.15.1), the RHS of Theorem 3.3.9 (2) applied to c is

$$(3.3.18.1) \quad \sum_{\substack{j < 0, J \\ k, K \cup J = N}} (-1)^{(N-K)(a+N-J)} \sigma(N \setminus J, N \setminus K) Z^{-2-j-k|N \setminus (J \cap K)} a_{(j|J)} (b_{(k|K)} c) + \\ + \sum_{\substack{j \geq 0, J \\ k, K \cup J = N}} (-1)^{(N-J)(b+N-K)+ab} \sigma(N \setminus K, N \setminus J) Z^{-2-j-k|N \setminus (J \cap K)} b_{(k|K)} (a_{(j|J)} c).$$

To compute the constant coefficient in the last formula, we let $K = J = N$, and $k = -2 - j$, to get

$$\sum_{j \geq -1} a_{(-2-j|N)} (b_{(j|N)} c) + (-1)^{ab} \sum_{j \geq 0} b_{(-2-j|N)} (a_{(j|N)} c).$$

Noting that the term with $j = -1$ in the first summand in the last formula is $: a : bc ::$, the theorem follows. \square

We thus arrive to the following equivalent definition of an $N_W = N$ SUSY vertex algebra (cf. [BK03]):

Definition 3.3.19. An $N_W = N$ SUSY vertex algebra is a tuple $(V, T, S^i, [\cdot_\Lambda \cdot], |0\rangle, ::)$, where

- $(V, T, S^i, [\cdot_\Lambda \cdot])$ is an $N_W = N$ SUSY Lie conformal algebra,
- $(V, |0\rangle, T, S^i, ::)$ is a unital quasicommutative quasiassociative differential superalgebra (i.e. T is an even derivation of $::$ and S^i ($i = 1, \dots, N$) are odd derivations of $::$),
- the Λ -bracket and the product $::$ are related by the non-commutative Wick formula (3.2.21.1).

Proof. We have shown that this definition follows from Definition 3.3.1. For the converse, we refer the reader to [BK03]. The proof carries over to the SUSY case with minor modifications. \square

Removing the “quantum corrections” we arrive to the following definition:

Definition 3.3.20. An $N_W = N$ Poisson SUSY vertex algebra is tuple $(V, |0\rangle, T, S^i, \{\cdot_\Lambda \cdot\}, \cdot)$, where

- $(V, T, S^i, \{\cdot_\Lambda \cdot\})$ is an $N_W = N$ SUSY Lie conformal algebra,
- $(V, |0\rangle, T, S^i, \cdot)$ is an unital commutative associative differential superalgebra,

- the following *Leibniz rule* is satisfied:

$$\{a_{\Lambda}bc\} = \{a_{\Lambda}b\}c + (-1)^{(a+N)b}b\{a_{\Lambda}c\}.$$

Theorem 3.3.21. *Let V be an $N_W = N$ SUSY vertex algebra. For each $a, b \in V$, $k \in \mathbb{Z}$ and $K \subset \{1, \dots, N\}$, the following identity, called Borchers identity, holds:*

$$(3.3.21.1) \quad \begin{aligned} & \left(i_{z,w}(Z-W)^{k|K} \right) Y(a, Z)Y(b, W) - (-1)^{ab} \left(i_{w,z}(Z-W)^{k|K} \right) Y(b, W)Y(a, Z) = \\ & = \sum_{j \geq 0, J} \sigma(J, K) \sigma(J \cup K) \left(\partial_W^{(j|J)} \delta(Z, W) \right) Y(a_{(k+j|K \cup J)} b, W). \end{aligned}$$

Proof. The LHS of (3.3.21.1) is local since multiplied by $(z-w)^n$ for $n \gg 0$ it is equal to

$$(Z-W)^{n+k|K} [Y(a, Z), Y(b, W)] = 0,$$

by the locality axiom. Therefore we can apply the decomposition Lemma 3.1.7 to the LHS of (3.3.21.1). We have

$$\begin{aligned} c_{j|J}(W) &= \sigma(J, K) \operatorname{res}_Z \left(\left(i_{z,w}(Z-W)^{k+J|K \cup J} \right) Y(a, Z)Y(b, W) - \right. \\ & \quad \left. - (-1)^{ab} \sigma(J, K) \left(i_{w,z}(Z-W)^{k+j|K \cup J} \right) Y(b, W)Y(a, Z) \right), \end{aligned}$$

therefore the theorem follows from (3.2.18.3) and Theorem 3.3.9 (1). \square

Proposition 3.3.22. *Let V be a $N_W = N$ SUSY vertex algebra. Then*

$$(3.3.22.1) \quad [a_{(n|I)}, Y(b, W)] = \sum_{(j|J): j \geq 0} (-1)^{JN+IN+IJ} \sigma(J) \sigma(I) \times \\ \times \left(\partial_W^{(j|J)} W^{n|I} \right) Y(a_{(j|J)} b, W).$$

If, moreover, $n \geq 0$, this becomes:

$$(3.3.22.2) \quad [a_{(n|I)}, Y(b, W)] = Y(e^{-W \nabla} a_{(n|I)} e^{W \nabla} b, W).$$

Proof. Multiplying the OPE formula (3.3.9.1) by $Z^{n|I}$ and taking residues we obtain in the left hand side $\sigma(I)[a_{(n|I)}, Y(b, W)]$, while the right hand side is

$$\begin{aligned} & \operatorname{res}_Z \sum_{(j|J): j \geq 0} (-1)^{I(N-J)} \sigma(J) (\partial_W^{(j|J)} \delta(Z, W) Z^{n|I}) Y(a_{(j|J)} b, W) = \\ & = \operatorname{res}_Z \sum_{(j|J): j \geq 0} (-1)^{I(N-J)} \sigma(J) (\partial_W^{(j|J)} \delta(Z, W) W^{n|I}) Y(a_{(j|J)} b, W) = \\ & = \sum_{(j|J): j \geq 0} (-1)^{I(N-J)+JN} \sigma(J) (\partial_W^{(j|J)} W^{n|I}) Y(a_{(j|J)} b, W) \end{aligned}$$

hence (3.3.22.1) follows. Note that when $n \geq 0$, the RHS of (3.3.22.1) is a finite sum of fields of V times monomials $W^{k|K}$ with $k \geq 0$. This easily implies that this field is local with respect to all fields of V , hence the LHS of (3.3.22.2) is local with respect to all fields of V . On the other hand, the adjoint action of ∇ on $a_{(n|I)}$ either decreases n or $\sharp I$ (cf. (3.3.2.5)). Using the formula $\operatorname{Ad} e^X = e^{\operatorname{ad} X}$ for an

even element X of a Lie superalgebra, we get that $e^{-W^\nabla} a_{(n|I)} e^{W^\nabla}$ is a finite sum, involving only positive powers of w , hence the RHS of (3.3.22.2) is also local with respect to all fields of V .

To apply the uniqueness theorem, we need to check that both sides agree when valuated at the vacuum vector. The left hand side is given by

$$[a_{(n|I)}, Y(b, W)]|0\rangle = a_{(n|I)} e^{W^\nabla} b,$$

where we used the fact that $a_{(n|I)}|0\rangle = 0$ and Proposition 3.3.7 (1). On the other hand, by the same proposition the left hand side is

$$Y(e^{-W^\nabla} a_{(n|I)} e^{W^\nabla}, W)|0\rangle = e^{W^\nabla} e^{-W^\nabla} a_{(n|I)} e^{W^\nabla} |0\rangle,$$

and (3.3.22.2) follows. \square

Remark 3.3.23. As a consequence of (3.3.22.1) we see that by taking the coefficient of $W^{-1-k|N\setminus K}$ we obtain the commutator $[a_{(n|I)}, b_{(k|K)}]$ as a linear combination of Fourier modes of fields in V . This rather complicated formula says that the linear span of Fourier modes of $\text{End}(V)$ -valued fields is a Lie superalgebra. In order to compute explicitly the Lie bracket, we compute the coefficient of $W^{-1-k|N\setminus K}$ on the left hand side of (3.3.22.1) to obtain:

$$(3.3.23.1) \quad (-1)^{(a+N-I)(N-K)} [a_{(n|I)}, b_{(k|K)}].$$

To compute this coefficient on the right hand side we first expand:

$$(3.3.23.2) \quad (\partial_W^{j|J} W^{n|I}) W^{-1-l|N\setminus L} = \frac{(-1)^{\frac{J(J-1)}{2}} n!}{(n-j)!} \times \\ \times \sigma(J, I \setminus J) \sigma(I \setminus J, N \setminus L) W^{n-j-1-l|(I \setminus J) \cup (N \setminus L)}.$$

Note that in order for the corresponding term in (3.3.22.1) not to vanish, we must have $J \subset I$ and in order for the coefficient of $W^{-1-k|N\setminus K}$ not to be zero in (3.3.23.2) we must have $(K \cap I) \subset J$. Now we set $n-j-l-1 = -1-k$ and $(I \setminus J) \cup (N \setminus L) = N \setminus K$ to obtain $l = n+k-j$ and $L = K \cup (I \setminus J)$. We get then for the right hand side

$$\sum_{(j|J): j \geq 0} (-1)^{(J+I)(N-J)} \binom{n}{j} \sigma(J) \sigma(I) \times \\ \times \sigma(J, I \setminus J) \sigma(I \setminus J, (N \setminus K) \setminus (I \setminus J)) (a_{(j|J)} b)_{(n+k-j|K \cup (I \setminus J))}$$

Combining with (3.3.23.1), we obtain:

$$(3.3.23.3) \quad [a_{(n|I)}, b_{(k|K)}] = (-1)^{(a+N-I)(N-K)} \sum_{(j|J): j \geq 0} (-1)^{(I-J)(N-J)} \binom{n}{j} \sigma(J) \times \\ \times \sigma(I) \sigma(J, I \setminus J) \sigma(I \setminus J, (N \setminus K) \setminus (I \setminus J)) (a_{(j|J)} b)_{(n+k-j|K \cup (I \setminus J))}.$$

3.3.24. We can define the *tensor product* of two $N_W = N$ SUSY vertex algebras in the usual way, namely, let V and W be two $N_W = N$ SUSY vertex algebras. The space of states is the vector superspace $V \otimes W$. The vacuum vector is $|0\rangle_V \otimes |0\rangle_W$.

Let us denote Y_V and Y_W the corresponding state-field correspondences. We define the state field correspondence Y for $V \otimes W$ as

$$(3.3.24.1) \quad Y(a \otimes b, Z) = Y_V(a, Z) \otimes Y_W(b, Z) = \\ = \sum_{(j|J), (k|K)} (-1)^{a(N-K)} \sigma(N \setminus K, N \setminus J) Z^{-2-j-k|(N \setminus (J \cap K))} a_{(j|J)} \otimes b_{(k|K)},$$

where the endomorphism $a_{(j|J)} \otimes b_{(k|K)}$ is defined to be

$$(a_{(j|J)} \otimes b_{(k|K)})(v \otimes w) = (-1)^{(b+N-K)v} a_{(j|J)} v \otimes b_{(k|K)} w.$$

Note that in order for σ not to vanish in (3.3.24.1) we must have $J \cup K = N$. Finally, we let the translation operators be $T = T_V \otimes Id + Id \otimes T_W$ and $S^i = S_V^i \otimes Id + Id \otimes S_W^i$. All the axioms of SUSY vertex algebra are straightforward to check.

Theorem 3.3.25 (Existence). *Let V be a vector superspace, $|0\rangle \in V$ an even vector, T an even endomorphism of V and S^i , $i = 1, \dots, N$, odd endomorphisms of V , pairwise anticommuting between themselves and commuting with T . Suppose moreover that $T|0\rangle = S^i|0\rangle = 0$. Let \mathcal{F} be a family of $\text{End}(V)$ -valued fields*

$$a^\alpha(Z) = \sum_{j \in \mathbb{Z}, J} Z^{-1-j|N \setminus J} a_{(j|J)}^\alpha$$

indexed by $\alpha \in A$, such that

- (1) $a^\alpha(Z)|0\rangle|_{Z=0} = a^\alpha \in V$,
- (2) $[T, a^\alpha(Z)] = \partial_z a^\alpha(Z)$ and $[S^i, a^\alpha(Z)] = \partial_{\theta^i} a^\alpha(Z)$,
- (3) all pairs $(a^\alpha(Z), a^\beta(Z))$ are local,
- (4) the vectors $a_{(j_s|J_s)}^{\alpha_s} \dots a_{(j_1|J_1)}^{\alpha_1} |0\rangle$ span V .

Then the formula

$$(3.3.25.1) \quad Y(a_{(j_s|J_s)}^{\alpha_s} \dots a_{(j_1|J_1)}^{\alpha_1} |0\rangle, Z) = \\ = \prod \sigma(J_i) a^{\alpha_s}(Z)_{(j_s|J_s)} \left(\dots a_{(j_2|J_2)}^{\alpha_2} (a_{(j_1|J_1)}^{\alpha_1} \text{Id}) \dots \right)$$

gives a well defined structure of an $N_W = N$ SUSY vertex algebra on V , with vacuum vector $|0\rangle$, translation operators T , S^i , and such that $Y(a^\alpha, Z) = a^\alpha(Z)$.

Such a structure is unique.

Proof. Let $\tilde{\mathcal{F}}$ be the minimal family of $\text{End}(V)$ -valued fields containing \mathcal{F} , closed under all $(j|J)$ -products and under the derivations ∂_z and ∂_{θ^i} , and subject to the conditions (1)-(3) of the Theorem. By Theorem 3.3.3, $\tilde{\mathcal{F}}$ is an $N_W = N$ SUSY vertex algebra. Define a map $\varphi : \tilde{\mathcal{F}} \rightarrow V$ by $a(Z) \mapsto a(Z)|0\rangle_{Z=0}$. This map is surjective by (4). Let $a(Z) \in \ker \varphi$. It follows easily by (2) and the fact that $T|0\rangle = S^i|0\rangle = 0$ that $a(Z)|0\rangle = 0$. Since φ is surjective, for each $b \in V$, there exists $b(W) \in \varphi^{-1}(b)$. By (3) there exists $j \in \mathbb{Z}_+$ such that

$$(z-w)^j a(Z) b(W) |0\rangle = (-1)^{ab} (z-w)^j b(W) a(Z) |0\rangle = 0$$

Letting $W = 0$ and canceling z^j we obtain $a(Z)b = 0$ for every $b \in V$, hence $a(Z) = 0$ and φ is also injective. We obtain thus a *state-field correspondence* $Y : a \mapsto Y(a, Z)$. Formula (3.3.25.1) follows from the $(j|J)$ -product identity in Theorem 3.3.9 (1). \square

3.4. The universal enveloping SUSY vertex algebra. In this section we construct maps φ and φ' , used in [Hel06] to define the conformal blocks, and we construct an $N_W = N$ SUSY vertex algebra attached to each $N_W = N$ SUSY Lie conformal algebra.

3.4.1. Let V be an $N_W = N$ SUSY vertex algebra. According to Theorem 3.3.17 it is a SUSY Lie conformal algebra. It follows by Proposition 3.2.12 that the pair $(\text{Lie}(V), V)$ is an $N_W = N$ formal distribution Lie superalgebra.

Recall from Lemma 3.2.9 and 3.2.10 the construction of the Lie superalgebra $\text{Lie}(V) = \tilde{V}/\tilde{\nabla}\tilde{V}$, where $\tilde{V} = V \otimes_{\mathbb{C}} \mathbb{C}[X, X^{-1}]$ and $\tilde{\nabla}\tilde{V}$ is the space spanned by vectors of the form:

$$(3.4.1.1) \quad Ta \otimes f(X) + a \otimes \partial_x f(X), \quad S^i a \otimes f(X) + (-1)^{a_N} a \otimes \partial_{\eta^i} f(X),$$

for $a \in V$, $f(X) \in \mathbb{C}[X, X^{-1}]$, and we change the parity if N is odd.

Let $\varphi : \text{Lie}(V) \rightarrow \text{End}(V)$ be the linear map defined by

$$(3.4.1.2) \quad a_{<n|I>} = a \otimes X^{n|I} \mapsto (-1)^{a_I} \sigma(I) a_{(n|I)}, \quad a \in V.$$

Similarly, we construct $V \otimes_{\mathbb{C}} \mathbb{C}((X))$ and consider its quotient $\text{Lie}'(V)$ by the vector space generated by vectors of the form (3.4.1.1), with reversed parity if N is odd. Then (3.4.1.2) defines a map $\varphi' : \text{Lie}'(V) \rightarrow \text{End}(V)$. Comparing (3.2.10.3) and (3.3.23.3) and noting the extra factor $\sigma(J)$ in (3.3.17.1) we obtain the following

Theorem 3.4.2. *The maps φ , and φ' are Lie superalgebra homomorphisms.*

3.4.3. Let \mathcal{R} be an $N_W = N$ SUSY Lie conformal algebra, and let $(\text{Lie}(\mathcal{R}), \mathcal{R})$ be the corresponding $N_W = N$ formal distribution Lie superalgebra (cf. Proposition 3.2.12). The Lie bracket in $\text{Lie}(\mathcal{R})$ is given by (3.2.10.3). Recall from 3.2.13 that $\text{Lie}(\mathcal{R})$ is a regular $N_W = N$ formal distribution Lie superalgebra. In particular, it carries an even derivation T and N odd derivations S^i , $i = 1, \dots, N$ defined by (3.2.13.1). Moreover, the annihilation subalgebra $\text{Lie}(\mathcal{R})_{\leq}$ is invariant by these derivations.

Theorem 3.4.4. *Let \mathcal{R} be an $N_W = N$ SUSY Lie conformal algebra. Let $V = V(\mathcal{R})$ be the quotient of $U(\text{Lie}(\mathcal{R}))$ by the left ideal generated by $\text{Lie}(\mathcal{R})_{\leq}$. Then V admits an $N_W = N$ SUSY vertex algebra structure whose vacuum vector $|0\rangle$ is the image of 1 in V , and the translation operators T , S^i ($i = 1, \dots, N$), are obtained by extending the corresponding derivations on $\text{Lie}(\mathcal{R})$ by the Leibniz rule. This vertex algebra is called the universal enveloping vertex algebra of \mathcal{R} .*

Proof. Let \mathcal{F} be the family of $\text{Lie}(\mathcal{R})$ -valued formal distributions

$$\mathcal{F} = \left\{ a(Z) \middle| a \in \mathcal{R} \right\},$$

where $a(Z)$ was defined in (3.2.10.1). Note that this family defines a family of $\text{End}(V)$ -valued formal distributions, where the action is by left multiplication. This family satisfies (1)-(4) of Theorem 3.3.25. Indeed, (2) follows from (3.2.13.2), (3) follows from Proposition 3.2.11 and (4) follows since the vectors $a_{(j|J)}$ with $a \in \mathcal{R}$, $j \in \mathbb{Z}$ and $J \subset \{1, \dots, N\}$ span $\text{Lie}(\mathcal{R})$. The theorem will follow from the existence Theorem 3.3.25 if we show that these distributions are in fact $\text{End}(V)$ -valued fields.

For that, given $a^{\alpha_1}, \dots, a^{\alpha_s} \in \mathcal{R}$, we have to prove that for any $a \in \mathcal{R}$ and $J \subset \{1, \dots, N\}$, we have: $a_{(n|J)} a_{(j_1|J_1)}^{\alpha_1} \dots a_{(j_s|J_s)}^{\alpha_s} |0\rangle = 0$ for $n \gg 0$. This is proved by induction on s , using (3.2.10.3). \square

4. STRUCTURE THEORY OF $N_K = N$ SUSY VERTEX ALGEBRAS

4.1. In this section we develop the structure theory of $N_K = N$ SUSY vertex algebras, where N is a positive integer. This kind of structures has been studied, in some particular cases, in the physics literature. Roughly speaking an $N_K = N$ SUSY vertex algebra is an $N_W = N$ SUSY vertex algebra, where the differential operators ∂_{θ^i} are replaced by the differential operators

$$D_Z^i = D_Z^{e_i} = \partial_{\theta^i} + \theta^i \partial_z.$$

To describe the corresponding SUSY Lie conformal superalgebras, perhaps the language of H -pseudoalgebras is more convenient [BDK01]. On the other hand, we are interested in their universal enveloping vertex algebras and in particular we want a description along the lines of the previous sections.

In order to have a uniform notation between this section and the previous ones, given two sets of coordinates $Z = (z, \theta^i)$ and $W = (w, \zeta^i)$ we will denote

$$(4.1.1) \quad \begin{aligned} Z - W &= \left(z - w - \sum_{i=1}^N \theta^i \zeta^i, \theta^j - \zeta^j \right), \\ (\theta - \zeta)^J &= \prod_{i \in J} (\theta^i - \zeta^i), \quad (Z - W)^{j|J} = \left(z - w - \sum_{i=1}^N \theta^i \zeta^i \right)^j (\theta - \zeta)^J, \end{aligned}$$

where $j \in \mathbb{Z}$ and J is an ordered subset of $\{1, \dots, N\}$. As before, we define

$$Z^{j|J} = z^j \theta^J.$$

Note that

$$(4.1.2) \quad (Z - W)^{-1|0} = \sum_{k=0}^N \frac{\left(\sum_{i=1}^N \theta^i \zeta^i \right)^k}{(z - w)^{k+1}},$$

therefore $(Z - W)^{-1|N}$ coincides with that in the $N_W = N$ case:

$$(4.1.3) \quad (Z - W)^{-1|N} = \frac{(\theta - \zeta)^N}{z - w}.$$

The differential operators D_Z^i satisfy the commutation relations

$$(4.1.4) \quad [D_Z^i, D_Z^j] = 2\delta_{i,j} \partial_z,$$

and, as before, we denote for $J = (j_1, \dots, j_k)$:

$$(4.1.5) \quad D_Z = (\partial_z, D_Z^1, \dots, D_Z^N), \quad D_Z^{j|J} = \partial_z^j D_Z^{j_1} \dots D_Z^{j_k}, \quad D_Z^{(j|J)} = \frac{(-1)^{\frac{j(j+1)}{2}}}{j!} D_Z^{j|J}.$$

Occasionally, when $j = 0$, we will write $D_Z^{0|J} = D_Z^J$.

Finally, in this section we will consider not necessarily disjoint subsets $I, J \subset \{1, \dots, N\}$ as in the $N_W = N$ case. Given I and J , ordered subsets of $\{1, \dots, N\}$, we will write $I \triangle J = (I \setminus J) \cup (J \setminus I)$. Note, however, that still $(Z - W)^{j|J} (Z - W)^{k|K} = 0$ if $J \cap K \neq \emptyset$. We will use the same formal δ -function $\delta(Z, W)$ as before. Remarkably, the new binomial $(Z - W)^{j|J}$, given by (4.1.1) “behaves” with respect to the operators $D_W^{j|J}$, in the same way as the old binomials (3.1.3.1) with respect to $\partial_W^{j|J}$.

Lemma 4.2. *The following identity is true:*

$$(4.2.1) \quad D_W^{(j|J)} \delta(Z, W) = \sigma(J)(i_{z,w} - i_{w,z})(Z - W)^{-1-j|N \setminus J}.$$

Proof. Let us assume for simplicity that $j = 0$, the general case follows easily from this, differentiating by w . We will prove the lemma by induction on $\sharp J$. When $J = \emptyset$, it follows from (4.1.3) that (4.2.1) coincides with the formula (1.6.1) for $\delta(Z, W)$. When $J = e_i = \{i\}$, the left hand side of (4.2.1) is given by

$$\begin{aligned} -D_W^i \delta(Z, W) &= -D_W^i(i_{z,w} - i_{w,z}) \frac{(\theta - \zeta)^N}{z - w} \\ &= -(i_{z,w} - i_{w,z}) \left(-\sigma(e_i) \frac{(\theta - \zeta)^{N \setminus e_i}}{z - w} + \zeta^i \frac{(\theta - \zeta)^N}{(z - w)^2} \right) \end{aligned}$$

On the other hand, using (4.1.2), we get:

$$\begin{aligned} (Z - W)^{-1|N \setminus e_i} &= \sum_{k \geq 0} \frac{(\sum \theta^i \zeta^i)^k}{(z - w)^{k+1}} (\theta - \zeta)^{N \setminus e_i} \\ &= \frac{(\theta - \zeta)^{N \setminus e_i}}{z - w} + \frac{\theta^i \zeta^i}{(z - w)^2} (\theta - \zeta)^{N \setminus e_i} \\ &= \frac{(\theta - \zeta)^{N \setminus e_i}}{z - w} - \sigma(e_i, N \setminus e_i) \frac{\zeta^i}{(z - w)^2} (\theta - \zeta)^N, \end{aligned}$$

hence (4.2.1) follows when $J = e_i$. To prove the general case, assume that the lemma is valid for $J = I \setminus e_i$. Since $D_W^I = \sigma(e_i, I \setminus e_i) D_W^i D_W^{I \setminus e_i}$ we have by the induction hypothesis

$$(4.2.2) \quad D_W^I \delta(Z, W) = \sigma(e_i, I \setminus e_i) \sigma(I \setminus e_i, N \setminus (I \setminus e_i)) (-1)^{\frac{(I-1)I}{2}} \times \\ \times D_W^i(i_{z,w} - i_{w,z})(Z - W)^{-1|N \setminus (I \setminus e_i)}.$$

We expand the last factor as:

$$(4.2.3) \quad D^i(Z - W)^{-1|N \setminus (I \setminus e_i)} = - \sum_{k \geq 1} k \zeta^i \frac{(\sum \theta^j \zeta^j)^{k-1}}{(z - w)^{k+1}} (\theta - \zeta)^{N \setminus (I \setminus e_i)} - \\ - \sigma(e_i, N \setminus I) \sum_{k \geq 0} \frac{(\sum \theta^j \zeta^j)^k}{(z - w)^{k+1}} (\theta - \zeta)^{N \setminus I} + \sum_{k \geq 0} (k+1) \zeta^i \frac{(\sum \theta^j \zeta^j)^k}{(z - w)^{k+2}} (\theta - \zeta)^{N \setminus (I \setminus e_i)}.$$

Relabeling the indexes we see that the first and last term cancel. Finally we note that, by (3.1.1.1):

$$(4.2.4) \quad \sigma(e_i, I \setminus e_i) \sigma(e_i, N \setminus I) \sigma(I \setminus e_i) = (-1)^I \sigma(I).$$

Combining (4.2.4), (4.2.3) and (4.2.2) we obtain the lemma for $j = 0$. \square

4.3. Most of the results proved in section 3 for the $N_W = N$ situation carry over to the $N_K = N$ setting with the following modifications.

- replace $\partial_Z = (\partial_z, \partial_{\theta^1}, \dots, \partial_{\theta^N})$ by $D_Z = (\partial_z, D_Z^1, \dots, D_Z^N)$,
- replace $Z - W = (z - w, \theta^i - \zeta^i)$ by $Z - W = \left(z - w - \sum_{i=1}^N \theta^i \zeta^i, \theta^j - \zeta^j \right)$,

- replace $(Z - W)^{j|J} = (z - w)^j \prod_{i \in J} (\theta^i - \zeta^i)$ by

$$(Z - W)^{j|J} = \left(z - w - \sum_{i=1}^N \theta^i \zeta^i \right)^j \prod_{i \in J} (\theta^i - \zeta^i),$$

- replace the commutative associative “translation” superalgebra $\mathbb{C}[T, S^i]$ by the non-commutative associative “translation” superalgebra \mathcal{H} generated by the set $\nabla = (T, S^1, \dots, S^N)$, where T is an even generator and S^i are odd generators, subject to the relations:

$$(4.3.1) \quad [T, S^i] = 0, \quad [S^i, S^j] = 2\delta_{ij}T;$$

- replace the commutative associative “parameter” superalgebra $\mathbb{C}[\lambda, \chi^i]$ by the non-commutative associative “parameter” superalgebra \mathcal{L} , generated by the set $\Lambda = (\lambda, \chi^1, \dots, \chi^N)$, where λ is an even generator and χ^i are odd generators, subject to the relations:

$$(4.3.2) \quad [\lambda, \chi^i] = 0, \quad [\chi^i, \chi^j] = -2\delta_{ij}\lambda;$$

Note that we have an isomorphism $\mathcal{H} \rightarrow \mathcal{L}$ given by $\nabla \mapsto -\Lambda$.

Lemma 4.4. *The formal δ -function satisfies the properties (1)-(7) of 3.1.6 after replacing ∂_W by D_W and writing $\Lambda + D_W = (\lambda + \partial_w, \chi^i + D^i)$.*

Proof. (1) is clear from Lemma 4.2. In order to prove (2) we use Lemma 4.2 to write:

$$\begin{aligned} (Z - W)^{j|J} D_W^{(n|I)} \delta(Z, W) &= \sigma(I) \sigma(J, N \setminus I) \times \\ &\quad \times (i_{z,w} - i_{w,z})(Z - W)^{-1-n+j|N \setminus (I \setminus J)}. \end{aligned}$$

Applying Lemma 4.2 to $D_W^{(n-j|I \setminus J)} \delta(Z, W)$ the result follows from the following property of σ , which follows from (3.1.1.1):

$$\sigma(J, N \setminus I) \sigma(I \setminus J) = \sigma(I) \sigma(I \setminus J, J).$$

Properties (3)-(7) are proved as in 3.1.6. \square

Lemma 4.5. $D_Z^i (Z - W)^{j|J} = \sigma(e_i, J \setminus e_i) (Z - W)^{j|J \setminus e_i} + j \sigma(e_i, J) (Z - W)^{j-1|J \cup e_i}$.

Proof. We prove the lemma by direct computation when $j \geq 0$:

$$\begin{aligned} D_Z^i (Z - W)^{j|J} &= (\partial_{\theta^i} + \theta^i \partial_z) \left(z - w - \sum \theta^i \zeta^i \right)^j (\theta - \zeta)^J \\ &= -j \zeta^i (Z - W)^{j-1|J} + \sigma(e_i, J \setminus e_i) (Z - W)^{j|J \setminus e_i} + \theta^i j (Z - W)^{j-1|J} \\ &= \sigma(e_i, J \setminus e_i) (Z - W)^{j|J \setminus e_i} + j \sigma(e_i, J) (Z - W)^{j-1|J \cup e_i}. \end{aligned}$$

When $j = -1$ we have

$$\begin{aligned}
D_Z^i(Z-W)^{-1|J} &= (\partial_{\theta^i} + \theta^i \partial_z) \sum_{k \geq 0} \frac{(\sum_i \theta^i \zeta^i)^k (\theta - \zeta)^J}{(z-w)^{k+1}} \\
&= \sum_{k \geq 0} k \zeta^i \frac{(\sum_i \theta^i \zeta^i)^{k-1} (\theta - \zeta)^J}{(z-w)^{k+1}} + \sigma(e_i, J \setminus e_i) \times \\
&\quad \times \sum_{k \geq 0} \frac{(\sum_i \theta^i \zeta^i)^k (\theta - \zeta)^{J \setminus e_i}}{(z-w)^{k+1}} - \theta^i \sum_{k \geq 0} (k+1) \frac{(\sum_i \theta^i \zeta^i)^k (\theta - \zeta)^J}{(z-w)^{k+2}} \\
&= \sigma(e_i, J \setminus e_i)(Z-W)^{-1|J \setminus e_i} - \sigma(e_i, J) \sum_{k \geq 0} (k+1) \frac{(\sum_i \theta^i \zeta^i)^k (\theta - \zeta)^{J \cup e_i}}{(z-w)^{k+2}} \\
&= \sigma(e_i, J \setminus e_i)(Z-W)^{-1|J \setminus e_i} - \sigma(e_i, J)(Z-W)^{-2|J \cup e_i}.
\end{aligned}$$

The general case follows from these by noting that $D_Z^{1|0} = (D_Z^i)^2 = \partial_z$, hence

$$(Z-W)^{-j-1|J} = \frac{1}{j!} (D_Z^i)^{2j} (Z-W)^{-1|J}.$$

Therefore we get for $j \geq 0$:

$$\begin{aligned}
D_Z^i(Z-W)^{-j-1|J} &= \frac{1}{j!} (D_Z^i)^{2j+1} (Z-W)^{-1|J} \\
&= \frac{1}{j!} (D_Z^i)^{2j} \left(\sigma(e_i, J \setminus e_i)(Z-W)^{-1|J} - \sigma(e_i, J)(Z-W)^{-2|J \cup e_i} \right) \\
&= \sigma(e_i, J \setminus e_i)(Z-W)^{-1-j|J \setminus e_i} - (j+1)\sigma(e_i, J)(Z-W)^{-j-2|J \cup e_i}.
\end{aligned}$$

□

The following decomposition lemma is now proved in the same manner as Lemma 3.1.7:

Lemma 4.6. *Let $a(Z, W)$ be a local distribution in two variables. Then $a(Z, W)$ can be uniquely decomposed in the following finite sum:*

$$a(Z, W) = \sum_{(j|J): j \geq 0} \left(D_W^{(j|J)} \delta(Z, W) \right) c_{j|J}(W).$$

The coefficients $c_{j|J}$ are given by

$$c_{j|J}(W) = \text{res}_Z (Z-W)^{j|J} a(Z, W).$$

Remark 4.7. Let $(Z-W)\Lambda = \left(z - w - \sum_{i=1}^N \theta^i \zeta^i \right) \lambda + \sum_{i=1}^N (\theta^i - \zeta^i) \chi^i$. Note that $-\partial_w$ and $-D_W^i$ satisfy the same commutation relations (4.3.2) as λ, χ^i , therefore $-\partial_w, -D_W^i$ generate an associative superalgebra isomorphic to \mathcal{L} . This allows us to consider \mathcal{L} as a module over \mathcal{H} , by letting D_W^i and ∂_w act as the following derivations of the superalgebra \mathcal{L} :

$$(4.7.1) \quad [D_W^i, \chi^j] = 2\delta_{ij}\lambda, \quad [\partial_w, \chi^i] = [\partial_w, \lambda] = [D_W^i, \lambda] = 0.$$

Lemma 4.8.

$$D_Z^i \exp((Z - W)\Lambda) = \chi^i \exp((Z - W)\Lambda) = -[D_W^i, \exp((Z - W)\Lambda)].$$

Proof. Since the exponent is a sum of non-commuting terms, the derivative of the exponential is not as obvious as in the $N_W = N$ case. Let $A = \sum_{j=1}^N (\theta^j - \zeta^j) \chi^j$. We have:

$$\begin{aligned} \exp((Z - W)\Lambda) &= \exp\left(\left(z - w - \sum_{j=1}^N \theta^j \zeta^j\right) \lambda\right) \exp(A), \\ \partial_{\theta^i} A^k &= \sum_{j=0}^{k-1} A^j \chi^i A^{k-j-1}. \end{aligned}$$

Since $[A, \chi^i] = -2(\theta^i - \zeta^i)\lambda$ we obtain

$$\partial_{\theta^i} A^k = \sum_{j=0}^{k-1} \chi^i A^{k-1} - 2j\lambda(\theta^i - \zeta^i) A^{k-2} = k\chi^i A^{k-1} - k(k-1)\lambda(\theta^i - \zeta^i) A^{k-2},$$

therefore

$$(4.8.1) \quad \partial_{\theta^i} \exp(A) = (\chi^i - \lambda(\theta^i - \zeta^i)) \exp(A)$$

from which the first equality of the lemma follows easily. The proof of the second equality of the lemma is similar. Note that from (4.7.1) we have:

$$[D_W^i, A] = -\chi^i - 2\lambda(\theta^i - \zeta^i),$$

from where it follows as in (4.8.1) that

$$[D_W^i, \exp(A)] = -(\chi^i + \lambda(\theta^i - \zeta^i)) \exp(A),$$

and the lemma follows by a straightforward computation. \square

4.9. Now we are in position to define the *formal Fourier transform* and $N_K = N$ *SUSY Lie conformal algebras* as we did in 3.1.8. We put

$$\mathcal{F}_{Z,W}^\Lambda a(Z, W) = \text{res}_Z \exp((Z - W)\Lambda) a(Z, W),$$

which formally looks exactly like (3.1.8.1) but in this expression the variables χ^i in $\Lambda = (\lambda, \chi^1, \dots, \chi^N)$ do not commute, but rather satisfy (4.3.2), and $(Z - W)$ is given by (4.1.1) instead of (3.1.3.1). Using this formal Fourier transform, we define the Λ -bracket of two formal distributions $a(W)$ and $b(W)$ as in (3.2.1.1). The $N_K = N$ version of Proposition 3.1.9, on the properties of the Fourier transform, is proved in the same way as in the $N_W = N$ case with the aid of Lemma 4.8. There is only one subtlety involved in stating and proving (3) of Proposition 3.1.9, and consequently, (3) in Proposition 3.2.2. Since the exponentials involved in this case do not commute, the argument in 3.1.9.2 is no longer valid. We define

$$\mathcal{F}_{X,W}^{\Lambda+\Gamma} = \mathcal{F}_{X,W}^\Psi|_{\Psi=\Lambda+\Gamma},$$

where the Fourier transform on the RHS is computed as follows. First compute $\mathcal{F}_{X,W}^\Psi$, and then replace Ψ by $\Lambda + \Gamma = (\lambda + \gamma, \chi^1 + \eta^1, \dots, \chi^N + \eta^N)$. Here Ψ denotes another set of indeterminates $\Psi = (\psi, v^1, \dots, v^N)$, where ψ is an even indeterminate and v^i are odd indeterminates, subject to the relations:

$$[\psi, v^i] = 0, \quad [v^i, v^j] = -2\delta_{ij}\psi.$$

We need to check that $\mathcal{F}_{Z,W}^\Lambda \mathcal{F}_{X,W}^\Gamma = (-1)^N \mathcal{F}_{X,W}^{\Lambda+\Gamma} \mathcal{F}_{Z,X}^\Lambda$, or, equivalently

$$(4.9.1) \quad \exp((Z-W)\Lambda) \exp((X-W)\Gamma) = \exp((X-W)\Psi) \exp((Z-W)\Lambda)|_{\Psi=\Lambda+\Gamma}.$$

In order to compare both sides in (4.9.1), we assume that Ψ and Λ commute :

$$(4.9.2) \quad [\lambda, \psi] = [\lambda, v^i] = 0, \quad [\chi^i, \psi] = [\chi^i, v^j] = 0.$$

Note that the RHS of (4.9.1) can be also computed as

$$(4.9.3) \quad \exp((X-W)(\Lambda+\Gamma)) \exp((Z-W)\Lambda),$$

where we have to use the following commutation relations between Λ and Γ :

$$(4.9.4) \quad [\eta^i, \chi^j] = 2\lambda\delta_{i,j}, \quad [\gamma, \chi^i] = [\gamma, \lambda] = [\lambda, \eta^i] = 0,$$

which follow from (4.9.2) after replacing Ψ by $\Lambda + \Gamma$.

In order to check (4.9.1), recall that, given two operators A, B , such that their commutator $[A, B] = C$ commutes with both A and B , we have

$$(4.9.5) \quad e^A e^B = e^C e^B e^A.$$

Let $Z = (z, \theta^i)$, $W = (w, \zeta^i)$ and $X = (x, \pi^i)$. Now we expand:

$$(4.9.6) \quad \exp((Z-W)\Lambda) \exp((X-W)\Gamma) = \exp\left((z-w-\sum \theta^i \zeta^i)\lambda\right) \times \\ \times \exp\left(\sum(\theta^i - \zeta^i)\chi^i\right) \exp\left((x-w-\sum \pi^i \zeta^i)\gamma\right) \exp\left(\sum(\pi^i - \zeta^i)\eta^i\right).$$

Note also that we have

$$(4.9.7) \quad \exp\left(\sum(\theta^i - \zeta^i)\chi^i\right) = \prod \exp((\theta^i - \zeta^i)\chi^i) = \prod (1 + (\theta^i - \zeta^i)\chi^i) \\ = \prod ((1 + \theta^i \chi^i)(1 - \zeta^i \chi^i)(1 + \theta^i \zeta^i \lambda)) = \prod e^{\theta^i \chi^i} e^{-\zeta^i \chi^i} e^{\theta^i \zeta^i \lambda} \\ = \exp\left(\sum \theta^i \chi^i\right) \exp\left(-\sum \zeta^i \chi^i\right) \exp\left(\sum \theta^i \zeta^i \lambda\right),$$

therefore (4.9.6) reads:

$$(4.9.8) \quad \exp((Z-W)\Lambda) \exp((X-W)\Gamma) = \exp((z-w)\lambda) \times \\ \times \exp\left(\sum \theta^i \chi^i\right) \exp\left(-\sum \zeta^i \chi^i\right) \exp((x-w)\gamma) \exp\left(\sum \pi^i \eta^i\right) \exp\left(-\sum \zeta^i \eta^i\right).$$

Commuting the exponentials using (4.9.5) and (4.9.4), (4.9.8) can be expressed as:

$$(4.9.9) \quad \exp((z-w)\lambda + (x-w)\gamma) \exp\left(-\sum \zeta^i \chi^i\right) \exp\left(\sum \pi^i \eta^i\right) \times \\ \times \exp\left(-\sum \zeta^i \eta^i\right) \exp\left(\sum \theta^i \chi^i\right) \exp\left(-2\sum \theta^i \pi^i \lambda\right).$$

Multiplying and dividing by $\exp(\sum \pi^i \chi^i)$ and using (4.9.7) we can express (4.9.9) as

$$\exp((z-w)\lambda + (x-w)\gamma) \exp\left(-\sum \zeta^i \chi^i\right) \exp\left(\sum \pi^i \eta^i\right) \times \\ \times \exp\left(-\sum \zeta^i \eta^i\right) \exp\left(\sum \pi^i \chi^i\right) \exp\left(\sum(\theta^i - \pi^i)\chi^i\right) \exp\left(-\sum \theta^i \pi^i \lambda\right).$$

Combining again the exponentials it is easy to express this as

$$\exp((z-w)\lambda + (x-w)\gamma) \exp\left(\sum(\pi^i - \zeta^i)(\chi^i + \eta^i)\right) \times \\ \times \exp\left(-\sum \pi^i \zeta^i(\lambda + \gamma)\right) \exp\left(\sum(\theta^i - \pi^i)\chi^i\right) \exp\left(-\sum \theta^i \pi^i \lambda\right),$$

which is equal to (4.9.3).

4.10. The definition of $(j|J)$ -th products for $j \geq 0$ and the definition of an $N_K = N$ formal distribution Lie superalgebra generalizes in a straightforward way the corresponding definitions in 3.2.1. The $N_K = N$ version of Proposition 3.2.2, on the properties of the Λ -bracket, is now proved in the same way as in the $N_W = N$ case, with the aid of (4.9.1).

Definition 4.11. An $N_K = N$ SUSY Lie conformal algebra is a $\mathbb{Z}/2\mathbb{Z}$ -graded \mathcal{H} -module \mathcal{R} , endowed with a parity $N \bmod 2$ \mathbb{C} -bilinear map $\mathcal{R} \otimes_{\mathbb{C}} \mathcal{R} \rightarrow \mathcal{L} \otimes_{\mathbb{C}} \mathcal{R}$ denoted (as before, we omit the symbol \otimes in the Λ -bracket)

$$a \otimes b \mapsto [a_{\Lambda} b] = \sum_{\substack{j \geq 0, J \\ \text{finite}}} (-1)^{JN} \Lambda^{(j|J)} a_{(j|J)} b,$$

where $a_{(j|J)} b \in \mathcal{R}$. This data should satisfy the following axioms:

(1) sesquilinearity (this is an equality in $\mathcal{L} \otimes \mathcal{R}$):

$$(4.11.1) \quad [S^i a_{\Lambda} b] = -(-1)^N \chi^i [a_{\Lambda} b], \quad [a_{\Lambda} S^i b] = (-1)^{a+N} (S^i + \chi^i) [a_{\Lambda} b],$$

where in the RHS of the second equation, to obtain an element of $\mathcal{L} \otimes \mathcal{R}$, we first compute the Λ -bracket, and then we commute S^i to the right using the relations $[S^i, \chi^j] = 2\delta_{ij}\lambda$ (cf. (4.7.1)).

(2) skew-symmetry (this is an equality in $\mathcal{L} \otimes \mathcal{R}$):

$$(4.11.2) \quad [a_{\Lambda} b] = -(-1)^{ab+N} [b_{-\Lambda-\nabla} a],$$

where the commutator on the right hand side is computed as follows: first compute $[b_{\Gamma} a] = \sum_{j \geq 0, J} \Gamma^{j|J} c_{j|J} \in \mathcal{L}' \otimes \mathcal{R}$, where \mathcal{L}' is another copy of \mathcal{L} generated by the set $\Gamma = (\gamma, \eta^1, \dots, \eta^N)$, where γ is an even generator, η^i are odd generators, subject to the relations

$$[\gamma, \eta^i] = 0, \quad [\eta^i, \eta^j] = -2\delta_{ij}\gamma.$$

Then replace Γ by $-\nabla - \Lambda = (-T - \lambda, -S^1 - \chi^1, \dots, -S^N - \chi^N)$ and apply T and S^i to $c_{j|J} \in \mathcal{R}$.

(3) Jacobi identity (this is an equality in $\mathcal{L} \otimes \mathcal{L}' \otimes \mathcal{R}$):

$$[a_{\Lambda} [b_{\Gamma} c]] = (-1)^{aN+N} [[a_{\Lambda} b]_{\Gamma+\Lambda} c] + (-1)^{(a+N)(b+N)} [b_{\Gamma} [a_{\Lambda} c]],$$

where $[[a_{\Lambda} b]_{\Gamma+\Lambda} c]$ is computed as follows, first compute $[[a_{\Lambda} b]_{\Psi} c] \in \mathcal{L} \otimes \mathcal{L}'' \otimes \mathcal{R}$, where \mathcal{L}'' is another copy of \mathcal{L} generated by the set $\Psi = (\psi, v^1, \dots, v^N)$, where ψ is an even generator, v^i are odd generators, subject to the relations

$$[\psi, v^i] = 0, \quad [v^i, v^j] = -2\delta_{ij}\psi.$$

Then replace Ψ by $\Lambda + \Gamma = (\lambda + \gamma, \theta^1 + \eta^1, \dots, \theta^N + \eta^N)$ to obtain an element of $\mathcal{L} \otimes \mathcal{L}' \otimes \mathcal{R}$.

It follows that given $N_K = N$ formal distribution Lie superalgebra $(\mathfrak{g}, \mathcal{R})$, the space \mathcal{R} is an $N_K = N$ SUSY Lie conformal algebra with $T = \partial_w$ and $S^i = D_W^i$.

Remark 4.12. We want to give an explanation for the commutation relations $[S^i, \chi^j] = 2\delta_{ij}\lambda$ appearing in sesquilinearity. For this, we give an abstract descriptions of the axioms of a Lie conformal algebra as follows. Let \mathcal{H} be a co-commutative Hopf superalgebra with commultiplication $\Delta : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$ and antipode S (note that

this is the case in definition 4.11). Let \mathcal{R} be a (left) \mathcal{H} -module. The spaces $\mathcal{R} \otimes \mathcal{R}$ and $\mathcal{H} \otimes \mathcal{R}$ are canonically \mathcal{H} modules with $h \mapsto \Delta h$ and we consider \mathcal{H} as a \mathcal{H} -module with the adjoint action. An \mathcal{H} -Lie conformal algebra structure in \mathcal{R} is a linear map $\phi : \mathcal{R} \otimes \mathcal{R} \rightarrow \mathcal{H} \otimes \mathcal{R}$ satisfying the following axioms (see [BDK01]):

- ϕ is an homomorphism of \mathcal{H} -modules, namely, the following diagram is commutative for any $h \in \mathcal{H}$:

$$\begin{array}{ccc} \mathcal{R} \otimes \mathcal{R} & \xrightarrow{\phi} & \mathcal{H} \otimes \mathcal{R} \\ \Delta h \downarrow & & \downarrow \Delta h \\ \mathcal{R} \otimes \mathcal{R} & \xrightarrow{\phi} & \mathcal{H} \otimes \mathcal{R} \end{array}$$

- (Sesquilinearity) Let L_h be the operator of left multiplication by h in \mathcal{H} . The following diagram is commutative:

$$\begin{array}{ccc} \mathcal{R} \otimes \mathcal{R} & \xrightarrow{\phi} & \mathcal{H} \otimes \mathcal{R} \\ h \otimes 1 \downarrow & & \downarrow L_h \otimes 1 \\ \mathcal{R} \otimes \mathcal{R} & \xrightarrow{\phi} & \mathcal{H} \otimes \mathcal{R} \end{array}$$

- (Skew-symmetry) Let A and B be two \mathcal{H} -modules. Let σ_{12} be the permutation isomorphism $A \otimes B \simeq B \otimes A$. Let $\mu : \mathcal{H} \otimes \mathcal{R} \rightarrow \mathcal{R}$ be the natural multiplication coming from the \mathcal{H} -module structure in \mathcal{R} . The following diagram is commutative:

$$\begin{array}{ccc} \mathcal{R} \otimes \mathcal{R} & \xrightarrow{\phi} & \mathcal{H} \otimes \mathcal{R} \\ \sigma_{12} \downarrow & & \downarrow (S \otimes \mu) \circ (\Delta \otimes 1) \\ \mathcal{R} \otimes \mathcal{R} & \xrightarrow{-\phi} & \mathcal{H} \otimes \mathcal{R} \end{array}$$

- (Jacobi identity) Define three morphisms $\mathcal{R}^{\otimes 3} \rightarrow \mathcal{H}^{\otimes 2} \otimes \mathcal{R}$ corresponding to the three terms in the Jacobi identity. First, let $\mu_{1\{23\}}$ be the composition

$$\mathcal{R}^{\otimes 3} \xrightarrow{1 \otimes \phi} \mathcal{R} \otimes \mathcal{H} \otimes \mathcal{R} \xrightarrow{\sigma_{12}(1 \otimes \phi)\sigma_{12}} \mathcal{H}^{\otimes 2} \otimes \mathcal{R}.$$

Similarly, we define $\mu_{2\{13\}}$ to be the composition:

$$\mathcal{R}^{\otimes 3} \xrightarrow{\sigma_{12}(1 \otimes \phi)\sigma_{12}} \mathcal{H} \otimes \mathcal{R}^{\otimes 2} \xrightarrow{(1 \otimes \phi)} \mathcal{H}^{\otimes 2} \otimes \mathcal{R}$$

Finally, let $\nu : \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H}$ be the multiplication map. We define $\mu_{\{12\}3}$ to be the composition:

$$\mathcal{R}^{\otimes 3} \xrightarrow{\phi \otimes 1} \mathcal{H} \otimes \mathcal{R}^{\otimes 2} \xrightarrow{1 \otimes \phi} \mathcal{H}^{\otimes 2} \otimes \mathcal{R} \xrightarrow{(\nu \otimes 1 \otimes 1)(1 \otimes \Delta \otimes 1)} \mathcal{H}^{\otimes 2} \otimes \mathcal{R}$$

The Jacobi identity is the following axiom:

$$\mu_{1\{23\}} = \mu_{\{12\}3} + \mu_{2\{13\}}$$

In the $N_K = N$ case, identifying:

$$S^i \mapsto -\chi^i, \quad T \mapsto -\lambda, \quad \gamma \mapsto \lambda, \quad \eta^i \mapsto \chi^i,$$

and, as in 3.2.7, changing the parity of \mathcal{R} if N is odd and defining

$$\phi(a \otimes b) = (-1)^{aN+N} [a_\Lambda b],$$

it is straightforward to check that the axioms of an $N_K = N$ SUSY Lie conformal algebra, as in Definition 4.11, get transformed into the axioms of an \mathcal{H} -Lie conformal algebra.

4.13. Lemmas 3.2.8 and 3.2.9 hold in the $N_K = N$ setting replacing ∂_W with D_W in (3.2.9.1). This allows us to construct a Lie superalgebra of degree $N \bmod 2$ $L(R) = \tilde{\mathcal{R}}/\tilde{\nabla}\tilde{\mathcal{R}}$, and the corresponding Lie superalgebra $\text{Lie}(\mathcal{R})$, from any $N_K = N$ SUSY Lie conformal algebra \mathcal{R} . In order to prove the $N_K = N$ versions of Propositions 3.2.11 and 3.2.12, we note that for $J = (j_1, \dots, j_k)$, we have

$$(4.13.1) \quad \begin{aligned} D_W^{j|J} &= \partial_w^j (\partial_{\zeta^{j_1}} + \zeta^{j_1} \partial_w) \dots (\partial_{\zeta^{j_k}} + \zeta^{j_k} \partial_w) \\ &= \sum_{K \subset J} \sigma(K, J \setminus K) \zeta^K \partial_W^{j+\#K|J \setminus K}. \end{aligned}$$

Let now $a_{<n|I>} = a \otimes W^{n|I} \in L(\mathcal{R})$ for each $a \in \mathcal{R}$. Using (4.13.1) and (3.2.9.1) with $f = W^{n|I}$, $g = W^{k|K}$ and letting $\Lambda = 0$, we compute the Lie bracket (of parity $N \bmod 2$) in $L(\mathcal{R})$:

$$(4.13.2) \quad \begin{aligned} \{a_{<n|I>}, b_{<k|K>}\} &= \sum_{j \geq 0, J} (-1)^{aJ+b(I-J)+\frac{(J \cap I)(J \cap I-1)}{2}+\frac{J(J-1)}{2}} \frac{(n)_{n-j-\#(J \setminus I)}}{j!} \times \\ &\times \sigma(J \setminus I, J \cap I) \sigma(J \cap I, I \setminus J) \sigma(J \setminus I, I \setminus J) \sigma(I \triangle J, K) (a_{(j|J)} b)_{<n+k-j-\#(J \setminus I)|K \cup (I \triangle J)>}, \end{aligned}$$

where $(n)_k = n(n-1) \dots (n-k+1)$. Defining $a_{(n|I)}$ as the image of $(-1)^{aI} \sigma(I) a_{<n|I>}$ in $\text{Lie}(\mathcal{R})$ and using (4.13.2) and Lemma 3.2.7 we compute the Lie bracket in $\text{Lie}(\mathcal{R})$:

$$(4.13.3) \quad \begin{aligned} [a_{(n|I)}, b_{(k|K)}] &= (-1)^{(a+N-I)(N-K)} \sum_{j \geq 0, K} (-1)^{J(N-I)+IN+\frac{(J \cap I)(J \cap I-1)}{2}+\frac{J(J+1)}{2}} \times \\ &\times \frac{(n)_{n-j-\#(J \setminus I)}}{j!} \sigma(I \triangle J, N \setminus (K \cup I \triangle J)) \sigma(I) \sigma(J \setminus I, J \cap I) \sigma(J \cap I, I \setminus J) \times \\ &\times \sigma(J \setminus I, I \setminus J) (a_{(j|J)} b)_{(n+k-j-\#(J \setminus I)|K \cup (I \triangle J))}. \end{aligned}$$

Substituting (3.2.11.2) in (4.13.1) we find:

$$(4.13.4) \quad \begin{aligned} D_W^{(j|J)} \delta(Z, W) &= \sum_{n \in \mathbb{Z}, I} (-1)^{\frac{(J \cap I)(J \cap I-1)}{2}+\frac{J(J+1)}{2}+I+(N-I)(J-I)} \times \\ &\times \frac{(n)_{n-j-\#(J \setminus I)}}{j!} \sigma(J \setminus I, J \cap I) \sigma(J \cap I) \times \\ &\times \sigma(I \setminus J, N \setminus I) \sigma(J \setminus I, I \setminus J) Z^{-1-n|N \setminus I} W^{n-j-\#(J \setminus I)|I \triangle J}. \end{aligned}$$

For each $a \in \mathcal{R}$ define the following $\text{Lie}(\mathcal{R})$ -valued formal distribution:

$$(4.13.5) \quad a(Z) = \sum_{n \in \mathbb{Z}, I} Z^{-1-n|N \setminus I} a_{(n|I)}.$$

Using (4.13.3) and (4.13.4) we obtain the $N_K = N$ analog of Proposition 3.2.11:

$$[a(Z), b(W)] = \sum_{j \geq 0, J} \left(D_W^{(j|J)} \delta(Z, W) \right) (a_{(j|J)} b)(W).$$

To prove the $N_K = N$ analog of Proposition 3.2.12 we need the following identity which is straightforward to check:

$$(S^i a)_{(j|J)} = \begin{cases} \sigma(e_i, N \setminus J) a_{(j|J \setminus e_i)} & \text{for } e_i \in J \\ -j\sigma(e_i, N \setminus (J \cup e_i)) a_{(j-1|J \cup e_i)} & \text{for } e_i \notin J. \end{cases}$$

4.14. As in 3.2.13, the $N_K = N$ formal distribution Lie superalgebra $\text{Lie}(\mathcal{R})$ carries an even derivation T and N odd derivations S^i ($i = 1, \dots, N$), given by:

$$\begin{aligned} T(a_{(j|J)}) &= -j a_{(j-1|J)}, \\ S^i(a_{(j|J)}) &= \begin{cases} \sigma(N \setminus J, e_i) a_{(j, J \setminus e_i)} & e_i \in J \\ j\sigma(N \setminus (J \cup e_i), e_i) a_{(j-1|J \cup e_i)} & e_i \notin J. \end{cases} \end{aligned}$$

It follows easily that the formal distributions (4.13.5) satisfy:

$$Ta(Z) = \partial_z a(Z), \quad S^i a(Z) = (\partial_{\theta^i} - \theta^i \partial_z) a(Z), \quad i = 1, \dots, N,$$

and therefore $(\text{Lie}(\mathcal{R}), \mathcal{R})$ is a *regular* $N_K = N$ formal distribution Lie superalgebra.

We define the normally ordered product of fields by the same formula (3.2.15.1) as in the $N_W = N$ case, and all the other products by using the derivations D_W instead of ∂_W . Lemma 3.2.17 is still valid in the $N_K = N$ setting (recall that $(Z - W)^{-1|N}$ is the same in both situations). The $N_K = N$ version of Proposition 3.2.19 is:

Proposition 4.15. *The following identities analogous to sesquilinearity for all pairs $(j|J)$ are true:*

$$\begin{aligned} (4.15.1) \quad (D_W^i a(W))_{(j|J)} b(W) &= -(-1)^J (\sigma(e_i, J) a(W)_{(j|J \setminus e_i)} b(W) + \\ &\quad + j\sigma(e_i, J) a(W)_{(j-1|J \cup e_i)} b(W)) \\ D_W^i (a(W)_{(j|J)} b(W)) &= (-1)^{N-J} \left((D_W^i a(W))_{(j|J)} b(W) + \right. \\ &\quad \left. + (-1)^a a(W)_{(j|J)} (D_W^i b(W)) \right). \end{aligned}$$

Proof. According to Lemma 4.5 we have:

$$\begin{aligned} (4.15.2) \quad \text{res}_Z i_{z,w}(Z - W)^{j|J} D_Z^i a(Z) b(W) &= \\ &= -(-1)^J \text{res}_Z \left(D_Z^i i_{z,w}(Z - W)^{j|J} \right) a(Z) b(W) = \\ &= -(-1)^J \text{res}_Z \left(\sigma(e_i, J \setminus e_i) i_{z,w}(Z - W)^{j|J \setminus e_i} + \right. \\ &\quad \left. + j\sigma(e_i, J) i_{z,w}(Z - W)^{j-1|J \cup e_i} \right) a(Z) b(W). \end{aligned}$$

Similarly we have:

$$\begin{aligned} (4.15.3) \quad -(-1)^{(a+1)b} \text{res}_Z i_{w,z}(Z - W)^{j|J} b(W) (D_Z^i a(Z)) &= \\ &= (-1)^{ab+J} \text{res}_Z \left(D_Z^i i_{w,z}(Z - W)^{j|J} \right) b(W) a(Z) = \\ &= (-1)^{ab+J} \text{res}_Z \left(\sigma(e_i, J \setminus e_i) i_{w,z}(Z - W)^{j|J \setminus e_i} + \right. \\ &\quad \left. + j\sigma(e_i, J) i_{w,z}(Z - W)^{j-1|J \cup e_i} \right) b(W) a(Z). \end{aligned}$$

Adding (4.15.2) and (4.15.3) we obtain:

$$\begin{aligned} (D_W^i a(W))_{(j|J)} b(W) &= -(-1)^J (\sigma(e_i, J \setminus e_i) a(W)_{(j|J \setminus e_i)} b(W) + \\ &\quad + j \sigma(e_i, J) a(W)_{(j-1|J \cup e_i)} b(W)). \end{aligned}$$

The fact that D_W^i is a derivation of all $(j|J)$ -products is proved in the same way as in (3.2.19.4):

$$\begin{aligned} D_W^i (a(W)_{(j|J)} b(W)) &= D_W^i \text{res}_Z \left(i_{z,w}(Z - W)^{j|J} a(Z) b(W) - \right. \\ &\quad \left. - (-1)^{ab} i_{w,z}(Z - W)^{j|J} b(W) a(Z) \right) = \\ (-1)^N \text{res}_Z \left(\left(-\sigma(e_i, J \setminus e_i) i_{z,w}(Z - W)^{j|J \setminus e_i} - j \sigma(e_i, J) i_{z,w}(Z - W)^{j-1|J \cup e_i} \right) \times \right. \\ &\quad \left. \times a(Z) b(W) + (-1)^{J+a} i_{z,w}(Z - W)^{j|J} a(Z) D_W^i b(W) + \right. \\ &\quad \left. + (-1)^{ab} \left(\sigma(e_i, J \setminus e_i) i_{w,z}(Z - W)^{j|J \setminus e_i} + j \sigma(e_i, J) i_{w,z}(Z - W)^{j-1|J \cup e_i} \right) b(W) a(Z) - \right. \\ &\quad \left. - (-1)^{ab+J} i_{w,z}(Z - W)^{j|J} D_W^i b(W) a(Z) \right) = \\ &= -(-1)^N \sigma(e_i, J \setminus e_i) a(W)_{(j|J \setminus e_i)} b(W) - (-1)^N j \sigma(e_i, J) a(W)_{(j-1|J \cup e_i)} b(W) + \\ &\quad + (-1)^{N+J+a} a(W)_{(j|J)} D_W^i b(W) = \\ &= (-1)^{N-J} \left((D_W^i a(W))_{(j|J)} b(W) + (-1)^a a(W)_{(j|J)} D_W^i b(W) \right) \end{aligned}$$

□

4.16. Even though the general Proposition 3.2.20 is no longer valid in the $N_K = N$ setting, we easily check that its proof works in the particular case $(j|J) = (-1|N)$. Therefore, the non-commutative Wick formula (3.2.21.1) is still valid in the $N_K = N$ case. The $N_K = N$ version of Dong's Lemma 3.2.22 is proved as in the usual vertex algebra case.

Definition 4.17. An $N_K = N$ SUSY vertex algebra is the data consisting of a super vector space V , an even vector $|0\rangle \in V$, N odd endomorphisms S^i and a parity preserving linear map Y from V to $\text{End}(V)$ -valued superfields $a \mapsto Y(a, Z)$, satisfying the following axioms:

- vacuum axioms:

$$Y(a, Z)|0\rangle = a + O(Z), \quad S^i|0\rangle = 0, \quad i = 1, \dots, N,$$

- translation invariance:

$$[S^i, Y(a, Z)] = \bar{D}_Z^i Y(a, Z),$$

where $\bar{D}_Z^i = \partial_{\theta^i} - \theta^i \partial_z$,

- locality:

$$(z - w)^n [Y(a, Z), Y(b, W)] = 0 \quad \text{for some } n \in \mathbb{Z}_+.$$

4.18. We define the $(j|J)$ -products for a $N_K = N$ SUSY vertex algebra, as in the $N_W = N$ case, by (3.3.2.2).

As in 3.3.2 we see easily that the vacuum axioms may be formulated as (3.3.2.4) and translation invariance is equivalent to:

$$(4.18.1) \quad [S^i, a_{(j,J)}] = \begin{cases} \sigma(N \setminus J, e_i) a_{(j, J \setminus e_i)} & e_i \in J \\ j\sigma(N \setminus (J \cup e_i), e_i) a_{(j-1|J \cup e_i)} & e_i \notin J \end{cases}$$

4.19. It follows easily from (4.18.1) and the vacuum axioms that

$$[S^i, S^j] = 2\delta_{i,j}T, \quad [S^i, T] = 0,$$

where T is an even operator satisfying:

$$[T, a_{(j|J)}] = -ja_{(j-1|J)} \quad \forall a, (j|J),$$

or equivalently:

$$[T, Y(a, Z)] = \partial_z Y(a, Z).$$

With these results we can prove the $N_K = N$ version of Theorem 3.3.3.

Theorem 4.20. *Let \mathcal{U} be a vector superspace and V a space of pairwise local $\text{End}(\mathcal{U})$ -valued fields such that V contains the constant field Id , it is invariant under the derivations $D_Z^i = \partial_{\theta^i} + \theta^i \partial_z$ and closed under all $(j|J)$ -th products. Then V is an $N_K = N$ SUSY vertex algebra with vacuum vector Id , the translation operators are $S^i a(Z) = D_Z^i a(Z)$, the $(j|J)$ product is the one for distributions multiplied by $\sigma(J)$.*

Proof. The proof goes like the proof of 3.3.3. To check translation invariance we see that $D_Z^i 1 = 0$ and that

$$\begin{aligned} \sigma(J) D_Z^i (a(Z)_{(j|J)} b(Z)) - (-1)^{a+N-J} a(Z)_{(j|J)} D_Z^i b(Z) &= \\ &= (-1)^{N-J} \sigma(J) (D_Z^i a(Z))_{(j|J)} b(Z). \end{aligned}$$

But in view of (4.15.1) this is:

$$\begin{aligned} &-(-1)^N \sigma(J) (\sigma(e_i, J) a(Z)_{(j|J \setminus e_i)} b(Z) + j\sigma(e_i, J) a(W)_{(j-1|J \cup e_i)} b(Z)) = \\ &= \sigma(N \setminus J, e_i) \sigma(J \setminus e_i) a(Z)_{(j|J \setminus e_i)} b(Z) + j\sigma(N \setminus (J \cup e_i), e_i) \sigma(J \cup e_i) a(Z)_{(j-1|J \cup e_i)} b(Z), \end{aligned}$$

proving equation (4.18.1). Locality is proved in the same way as in 3.3.3. \square

Lemma 3.3.5 is still valid for $N_K = N$ SUSY vertex algebras. Its proof parallels the proof for $N_W = N$ SUSY vertex algebras. Lemma 3.3.6, on the existence and uniqueness of solutions to a system of differential equations, is straightforward to generalize to the $N_K = N$ setting. The proof of Proposition 3.3.7 in this context is more subtle:

Proposition 4.21. *Let V be a $N_K = N$ SUSY vertex algebra. Then for every $a, b \in V$ we have:*

- (1) $Y(a, Z)|0\rangle = \exp(Z\nabla)a,$
- (2) $\exp(Z\nabla)Y(a, W)\exp(-Z\nabla) = i_{w,z}Y(a, W+Z),$
- (3) $Y(a, Z)_{(j|J)}Y(b, Z)|0\rangle = \sigma(J)Y(a_{(j|J)}b, Z)|0\rangle.$

where $\nabla = (T, S^1, \dots, S^N)$, $Z\nabla = zT + \sum \theta^i S^i$, and we define $W+Z = W - (-Z) = (z + w + \sum \zeta^i \theta^i, \theta^j + \zeta^j)^5$.

⁵Note that $Z + W \neq W + Z$

Proof. As in the proof of Proposition 3.3.7 we note that both sides of (1) and (3) are elements of $V[[Z]]$, whereas both sides of (2) are elements of $\text{End}(V)[[W, W^{-1}]][[Z]]$. By evaluating at $Z = 0$ we get equalities in all three cases. Indeed (1) and (2) are trivial, and (3) follows from the $N_K = N$ version of Lemma 3.3.5. We need to show that both sides in each equation satisfy the same system of differential equations.

(1) Similarly to the proof of Lemma 4.8, we expand:

$$(4.21.1) \quad \begin{aligned} \bar{D}_Z^i \exp(Z\nabla) &= (\partial_{\theta^i} - \theta^i \partial_z) \exp(zT) \sum_{k \geq 0} \frac{(\sum_i \theta^i S^i)^k}{k!} \\ &= (S^i + T\theta^i) \exp(Z\nabla) - \theta^i T \exp(Z\nabla) = S^i \exp(Z\nabla), \end{aligned}$$

from where the RHS $X(Z)$ of (1) satisfies the system of differential equations:

$$\bar{D}_Z^i X(Z) = S^i X(Z).$$

Similarly by translation invariance we have for the LHS of (1):

$$\bar{D}_Z^i Y(a, Z)|0\rangle = [S^i, Y(a, Z)]|0\rangle = S^i Y(a, Z)|0\rangle.$$

We also point out that a computation similar to (4.21.1) shows that

$$(4.21.2) \quad \bar{D}_Z^i \exp(-Z\nabla) = -\exp(-Z\nabla) S^i,$$

which is not entirely obvious since S^i does not commute with the exponential.

(2) By translation invariance we have:

$$\begin{aligned} \bar{D}_Z^i Y(a, W + Z) &= (-\zeta^i \partial_{w+z+\sum \zeta^i \theta^i} + \partial_{\zeta^i \theta^i} - \theta^i \partial_{w+z+\sum \zeta^i \theta^i}) Y(a, W + Z) = \\ &= \bar{D}_{W+Z}^i Y(a, Z + W) = [S^i, Y(a, Z + W)]. \end{aligned}$$

On the other hand, letting $Y(Z) = e^{Z\nabla} Y(a, W) e^{-Z\nabla}$ we have (cf. (4.21.2)):

$$\bar{D}_Z^i Y(Z) = S^i Y(Z) - (-1)^a Y(Z) S^i = [S^i, Y(Z)].$$

(3) For the RHS we have by translation invariance and the vacuum axioms:

$$S^i Y(a_{(j|J)} b, Z)|0\rangle = [S^i, Y(a_{(j|J)} b, Z)]|0\rangle = \bar{D}_Z^i Y(a_{(j|J)} b, Z)|0\rangle.$$

To prove that the LHS satisfies the same differential equation, we proceed exactly in the same way as in the proof of Proposition 3.3.7. We only need the fact that \bar{D}_Z^i is a derivation of all $(j|J)$ -products. But $\partial_z = (D_Z^i)^2$ is a derivation since:

$$\begin{aligned} \partial_z a(Z)_{(j|J)} b(Z) &= (-1)^{N-J} D_Z^i ((D_Z^i a(Z))_{(j|J)} b(Z) + (-1)^a a_{(j|J)} D_Z^i b(Z)) = \\ &= (\partial_z a(Z))_{(j|J)} b(Z) + (-1)^{a+1} (D_Z^i a(Z))_{(j|J)} D_Z^i b(Z) + \\ &\quad + (-1)^a (D_Z^i a(Z))_{(j|J)} D_Z^i b(Z) + a(Z)_{(j|J)} \partial_z b(Z) \end{aligned}$$

therefore $\bar{D}_Z^i = D_Z^i - 2\theta^i \partial_z$ is a derivation of all $(j|J)$ -products. \square

The uniqueness Proposition 3.3.8 is still valid in the $N_K = N$ setting, As its corollary, we obtain an analogous version of Theorem 3.3.9, namely

Theorem 4.22. *On an $N_K = N$ SUSY vertex algebra the following identities hold*

- (1) $Y(a_{(j|J)} b, Z) = \sigma(J) Y(a, Z)_{(j|J)} Y(b, Z)$.
- (2) $Y(a_{(-1|N)} b, Z) =: Y(a, Z) Y(b, Z) \cdot$
- (3) $Y(S^i a, Z) = D_Z^i Y(a, Z)$.

(4) We have the following OPE formula:

$$[Y(a, Z), Y(b, W)] = \sum_{(j|J): j \geq 0} \sigma(J)(D_W^{(j|J)} \delta(Z, W)) Y(a_{(j|J)} b, W)$$

Remark 4.23. Note that as a consequence of (3) we obtain

$$[S^i, Y(a, Z)] \neq Y(S^i a, Z),$$

in contrast to the $N_W = N$ and, in particular, the ordinary vertex algebra case.

Corollary 4.24.

$$\begin{aligned} (S^i a)_{(j|J)} &= \sigma(e_i, N \setminus J) a_{(j|J \setminus e_i)} - j \sigma(e_i, N \setminus (J \cup e_i)) a_{(j-1|J \cup e_i)} \\ &= \begin{cases} \sigma(e_i, N \setminus J) a_{(j|J \setminus e_i)} & \text{for } e_i \in J \\ -j \sigma(e_i, N \setminus (J \cup e_i)) a_{(j-1|J \cup e_i)} & \text{for } e_i \notin J, \end{cases} \\ S^i (a_{(j|J)} b) &= (-1)^{N-J} ((S^i a)_{(j|J)} b + (-1)^a a_{(j|J)} S^i b). \end{aligned}$$

4.25. In order to prove the $N_K = N$ version of the *associativity* formulas (3.3.12.3) and (3.3.12.4), we proceed as in 3.3.12, by taking the generating series of 4.22 (1) and using the following $N_K = N$ version of the *Taylor expansion*. For a formal power series $a(Z) \in \mathbb{C}[[Z]]$ we have:

$$(4.25.1) \quad a(W + Z) = \sum_{(j|J): j \geq 0} (-1)^{\frac{J(J-1)}{2}} \frac{W^{j|J}}{j!} D_Z^{j|J} a(Z) = e^{W D_Z} a(Z).$$

Indeed, the usual Taylor expansion is:

$$a(W + Z) = a\left(w + z + \sum \zeta^i \theta^i, \zeta^j + \theta^j\right) = \sum (-1)^{\frac{J(J-1)}{2}} \frac{\partial_W^{j|J}}{j!} a(W + Z)|_{W=0}.$$

In this case:

$$\begin{aligned} \partial_W^{1|0} a(W + Z)|_{W=0} &= D_Z^{1|0} a(Z), \\ \partial_W^{0|i} a(W + Z)|_{W=0} &= (\theta^i \partial_z + \partial_{\theta^i}) a(Z) = D_Z^i a(Z), \end{aligned}$$

proving (4.25.1).

Also, according to our prescription to add coordinates we see that

$$(X - Z) - W = X - (W + Z) = X - (W - (-Z)),$$

and note that equation (3.3.3.2) is still valid in the $N_K = N$ setting. The proof in 3.3.12 generalizes now easily. Similarly, we obtain as a corollary, the $N_K = N$ version of the Cousin property 3.3.13.

The proofs for *skew-symmetry* in Theorem 3.3.14 and quasi-commutativity for the normally ordered product as in 3.3.15 carry over verbatim to the $N_K = N$ case.

4.26. Defining as the *Fourier Transform* as in 3.3.16 we obtain an analogous result to Theorem 3.3.17, namely an $N_K = N$ SUSY vertex algebra gives rise to an $N_K = N$ SUSY Lie conformal algebra.

The $N_K = N$ version of *quasi-associativity* for the normally ordered product is the same and is proved in the same way as Theorem 3.3.18.

As in the $N_W = N$ case, we have the following equivalent definition of $N_K = N$ SUSY vertex algebras:

Definition 4.27. An $N_K = N$ SUSY vertex algebra is a tuple $(V, T, S^i, [\cdot_\Lambda \cdot], |0\rangle, ::)$, $i = 1, \dots, N$, where

- $(V, T, S^i, [\cdot_\Lambda \cdot])$ is an $N_K = N$ SUSY Lie conformal algebra,
- $(V, |0\rangle, T, S^i, ::)$ is a unital quasicommutative quasiassociative differential superalgebra (i.e. T is an even derivation of $::$ and S^i are odd derivations of $::$),
- the Λ -bracket and the product $::$ are related by the non-commutative Wick formula (3.2.21.1).

4.28. The definition of an $N_K = N$ Poisson SUSY vertex algebra is straightforward to generalize. Similarly, the $N_K = N$ version of Borchers identity in Theorem 3.3.21 and the $N_K = N$ version of the commutator formula in Proposition 3.3.22, are the same with ∂_W replaced by D_W and are proved in the same way as for $N_W = N$ SUSY vertex algebras. Following the argument in Remark 3.3.23 and using (4.13.1), we obtain the formula for the commutator of the Fourier coefficients of the $\text{End}(V)$ -valued fields of the $N_K = N$ SUSY vertex algebra: it is equal to the RHS of (4.13.3), multiplied by $\sigma(J)$.

The rest of section 3.3 carries over to the $N_K = N$ case with minor modifications, in particular we define tensor products of $N_K = N$ SUSY vertex algebras as in 3.3.24 and we have an *existence theorem* as in 3.3.25 that we restate here:

Theorem 4.29 (Existence of $N_K = N$ SUSY vertex algebras). *Let V be a vector space, $|0\rangle \in V$ an even vector, T an even endomorphism of V and S^i , $i = 1, \dots, N$ odd endomorphisms of V , satisfying $[S^i, S^j] = 2\delta_{i,j}T$. Suppose moreover that $S^i|0\rangle = 0$. Let \mathcal{F} be a family of fields $a^\alpha(Z) = \sum_{(j|J)} Z^{-1-j|N\setminus J} a_{(j|J)}^\alpha$, indexed by $\alpha \in A$, and such that*

- (1) $a^\alpha(Z)|0\rangle|_{Z=0} = a^\alpha \in V$,
- (2) $[S^i, a^\alpha(Z)] = \bar{D}_Z^i a^\alpha(Z)$,
- (3) all pairs $(a^\alpha(Z), a^\beta(Z))$ are local,
- (4) the vectors $a_{(j_s|J_s)}^{\alpha_s} \dots a_{(j_1|J_1)}^{\alpha_1} |0\rangle$ span V .

Then the formula

$$\begin{aligned} Y(a_{(j_s|J_s)}^{\alpha_s} \dots a_{(j_1|J_1)}^{\alpha_1} |0\rangle, Z) = \\ = \prod \sigma(J_i) a^{\alpha_s}(Z)_{(j_s|J_s)} \left(\dots a_{(j_2|J_2)}^{\alpha_2}(Z) (a_{(j_1|J_1)}^{\alpha_1}(Z) \text{Id}) \dots \right) \end{aligned}$$

defines a structure of an $N_K = N$ SUSY vertex algebra on V , with vacuum vector $|0\rangle$, translation operators S^i and such that $Y(a^\alpha, Z) = a^\alpha(Z)$, $\alpha \in A$.

Such a structure is unique.

4.30. The results in section 3.4 generalize to this context without difficulty. In particular, we obtain the *universal enveloping $N_K = N$ SUSY vertex algebra* of an $N_K = N$ SUSY Lie conformal algebra.

In the same way as in Theorem 3.4.2, we obtain:

Theorem 4.31. *Let V be an $N_K = N$ SUSY vertex algebra. Let $\mathcal{A} = \mathbb{C}[X, X^{-1}]$, define $L(V)$ to be the quotient of $\tilde{V} = \mathcal{A} \otimes_{\mathbb{C}} V$ by the span of vectors of the form:*

$$S^i a \otimes f(X) + (-1)^a a \otimes D_X^i f(X).$$

and let $L'(V)$ be its completion with respect to the natural topology in \mathcal{A} . Then $L(V)$ (resp. $L'(V)$) carries a natural Lie superalgebra of degree $N \bmod 2$ structure.

Let $\text{Lie}(V)$ (resp. $\text{Lie}'(V)$) be the corresponding Lie superalgebra. The map $\varphi : \text{Lie}(V) \rightarrow \text{End}(V)$ (resp. $\varphi' : \text{Lie}'(V) \rightarrow \text{End}(V)$), given by formula (3.4.1.2), is a Lie superalgebra homomorphism.

5. EXAMPLES

Example 5.1 ($V(\mathscr{W}_N)$ series). Let $X = (x, \xi^1, \dots, \xi^N)$, where x is even and ξ^i are odd anticommuting variables commuting with x . Consider the Lie algebra $\mathfrak{g} = W(1|N)$ of derivations of $\mathbb{C}[X, X^{-1}]$. It is spanned by elements of the form $X^{j|J}\partial_x$ and $X^{j|J}\partial_{\xi^i}$ (cf. Example 2.17). Define the following \mathfrak{g} -valued formal distributions:

$$(5.1.1) \quad L(Z) = -\delta(Z, X)\partial_x, \quad Q^i(Z) = -\delta(Z, X)\partial_{\xi^i}, \quad i = 1, \dots, N.$$

A straightforward computation shows that these distributions satisfy the following commutation relations:

$$(5.1.2) \quad \begin{aligned} [L(Z), L(W)] &= \delta(Z, W)\partial_w L(W) + 2(\partial_w \delta(Z, W))L(W), \\ [L(Z), Q^i(W)] &= \delta(Z, W)\partial_w Q^i(W) + (\partial_{\xi^i} \delta(Z, W))L(W) + \\ &\quad + (\partial_w \delta(Z, W))Q^i(W), \\ [Q^i(Z), Q^j(W)] &= \delta(Z, W)\partial_{\xi^i} Q^j(W) + (-1)^N (\partial_{\xi^i} \delta(Z, W))Q^j(W) - \\ &\quad - (-1)^N (\partial_{\xi^j} \delta(Z, W))Q^i(W). \end{aligned}$$

In particular, the distributions (5.1.1) are pairwise local. Let \mathscr{F} be the family of \mathfrak{g} -valued formal distributions

$$\mathscr{F} = \left\{ \partial_Z^{j|J} L(Z), \partial_Z^{j|J} Q^i(Z) \mid j \geq 0, J \subset \{1, \dots, N\}, i = 1, \dots, N \right\}.$$

Then $(\mathfrak{g}, \mathscr{F})$ is an $N_W = N$ SUSY formal distribution Lie superalgebra.

Let \mathscr{W}_N be the corresponding $N_W = N$ SUSY Lie conformal algebra. It is generated as a $\mathbb{C}[T, S^i]$ -module by a vector L of parity $N \bmod 2$ and N -vectors Q^i , $i = 1, \dots, N$, of parity $N+1 \bmod 2$ satisfying the following Λ -brackets (which follow from (5.1.2))

$$(5.1.3) \quad \begin{aligned} [L_\Lambda L] &= (T + 2\lambda)L, \quad [Q_\Lambda^i Q^j] = (S^i + \chi^i)Q^j - \chi^j Q^i, \\ [L_\Lambda Q^i] &= (T + \lambda)Q^i + (-1)^N \chi^i L. \end{aligned}$$

When $N = 0$, this is the centerless Virasoro conformal algebra, which admits a central extension defined by:

$$[L_\lambda L] = (T + 2\lambda)L + \frac{\lambda^3}{12}C$$

where C is even, central, and satisfies $TC = 0$.

Translating the formulas in [FK02], it follows that when $N = 1$, \mathscr{W}_1 admits a central extension of the form:

$$(5.1.4) \quad \begin{aligned} [L_\Lambda L] &= (T + 2\lambda)L, \quad [Q_\Lambda Q] = SQ + \frac{\lambda\chi}{3}C, \\ [L_\Lambda Q] &= (T + \lambda)Q - \chi L + \frac{\lambda^2}{6}C, \end{aligned}$$

where C is even, central, and satisfies $TC = SC = 0$.

When $N = 2$, \mathscr{W}_2 admits a central extension given by:

$$\begin{aligned} [L_\Lambda L] &= (T + 2\lambda)L, & [L_\Lambda Q^i] &= (T + \lambda)Q^i + \chi^i L, \\ [Q_\Lambda^i Q^i] &= S^i Q^i, & [Q_\Lambda^1 Q^2] &= (S^1 + \chi^1)Q^2 - \chi^2 Q^1 + \frac{\lambda}{6}C, \end{aligned}$$

where C is as above. It follows from [KvdL89], [FK02] that these algebras do not admit central extensions for $N \geq 3$.

If $N \leq 2$, we let $V(\mathscr{W}_N)$ be the universal enveloping $N_W = N$ SUSY vertex algebra of the central extension of \mathscr{W}_N as given by Theorem 3.4.4, and let $V(\mathscr{W}_N)^c$ be the quotient of $V(\mathscr{W}_N)$ by the ideal $(C - c|0\rangle)_{(-1|N)}V(\mathscr{W}_N)$, where $c \in \mathbb{C}$ is called the *central charge*.

When $N = 1$, expanding the superfields as

$$Q(Z) = -J(z) + \theta G^+(z), \quad L(Z) = G^-(z) + \theta \left(L(z) + \frac{1}{2} \partial_z J(z) \right),$$

we check that the fields L, J, G^\pm satisfy the commutation relations of the $N = 2$ vertex algebra as defined in Example 2.12.

When $N \geq 3$, we let $V(\mathscr{W}_N)$ be the universal enveloping $N_W = N$ SUSY vertex algebra of \mathscr{W}_N . It follows from the definitions that $\text{Lie}(\mathscr{W}_N) = W(1|N)$, the Lie superalgebra of derivations of $\mathbb{C}[X, X^{-1}]$. Also, $\text{Lie}(\mathscr{W}_N)_\leq = W(1|N)_\leq$ is the Lie superalgebra of derivations of $\mathbb{C}[X]$. Denote by $W(1|N)_< = \text{Lie}(\mathscr{W}_N)_< \subset \text{Lie}(\mathscr{W}_N)_\leq$ the Lie superalgebra of vector fields vanishing at the origin; it is spanned by vectors of the form $X^{j|J} \partial_x$ and $X^{j|J} \partial_{\xi^i}$, with $j + \sharp J > 0$.

Definition 5.2. An $N_W = N$ SUSY vertex algebra V is called *conformal* if there exists $N+1$ vectors $\nu, \tau^1, \dots, \tau^N$ in V such that their associated superfields $L(Z) = Y(\nu, Z)$ and $Q^i(Z) = Y(\tau^i, Z)$ satisfy (5.1.3) (or possibly a central extension) and moreover:

- $\nu_{(0|0)} = T$, $\tau_{(0|0)}^i = S^i$,
- the operator $\nu_{(1|0)}$ acts diagonally with eigenvalues bounded below and with finite dimensional eigenspaces.

If moreover, the action of $\text{Lie}(\mathscr{W}_N)_<$ on V can be exponentiated to the group of automorphisms of the disk $D^{1|N}$, we will say that V is *strongly conformal*. This amounts to the following extra condition:

- the operators $\nu_{(1|0)}$ and $\sum_{i=1}^N \sigma(e_i) \tau_{(0|e_i)}^i$ have integer eigenvalues.

If $a \in V$ is an eigenvector of the operator $\nu_{(1|0)}$ of eigenvalue Δ , we say that a has *conformal weight* Δ . This happens if a satisfies

$$[L_\Lambda a] = (T + \Delta\lambda)a + O(\lambda^2) + O(\chi^1, \dots, \chi^N),$$

If, moreover, a satisfies $[L_\Lambda a] = (T + \Delta\lambda)a$ we say that a is *primary*.

As in the ordinary vertex algebra case, the conformal weight $\Delta(a)$ is an important *book-keeping device*:

$$\Delta(Ta) = \Delta(a) + 1, \quad \Delta(S^i a) = \Delta(a), \quad \Delta(a_{(j|J)} b) = \Delta(a) + \Delta(b) - j - 1.$$

(Note that $\Delta(ab) = \Delta(a) + \Delta(b)$.) Furthermore, letting $\Delta(\chi^i) = 0$ and $\Delta(\lambda) = 1$, all terms in $[a_\Lambda b]$ have conformal weight $\Delta(a) + \Delta(b) - 1$.

Remark 5.3. It is clear from this definition that the $N_W = N$ SUSY vertex algebra $V(\mathscr{W}_N)^c$ defined in Example 5.1 is strongly conformal.

Example 5.4 (Free Fields). As an example of a strongly conformal $N_W = N$ SUSY vertex algebra, we will compute explicitly the free fields case, namely, let α, C be even vectors and let φ be an odd vector. Consider the $N_W = N$ SUSY Lie conformal algebra generated by these three vectors, where C is central and annihilated by ∇ , and the other commutation relations are:

$$[\alpha_\Lambda \varphi] = C.$$

Let \tilde{F}_N be its universal enveloping $N_W = N$ SUSY vertex algebra and F_N its quotient by the ideal $(C - |0\rangle)_{(-1|N)}\tilde{F}_N$.

Expanding the superfields

$$\alpha(Z) = a(z) + \theta\psi(z), \quad \varphi(Z) = \phi(z) + \theta b(z)$$

we find that the fields a, b, ψ and ϕ generate the well known $bc - \beta\gamma$ -system, namely, the non-trivial λ -brackets are (up to skew-symmetry):

$$[b_\lambda a] = [\psi_\lambda \phi] = 1.$$

When $N = 1$, this SUSY vertex algebra admits a $N_W = 1$ strongly conformal structure with:

$$\nu = \alpha_{(-2|1)}\varphi_{(-1|1)}|0\rangle, \quad \tau = -\alpha_{(-1|0)}\varphi_{(-1|1)}|0\rangle,$$

and central charge $c = 3$. The associated fields $L = Y(\nu, Z)$ and $Q = Y(\tau, Z)$ are:

$$L = : (T\alpha)\varphi : \quad Q = - : (S\alpha)\varphi : .$$

In order to check the commutation relations (5.1.4) we use the non-commutative Wick formula (3.2.21.1) to find:

$$[\alpha_\Lambda L] = T\alpha, \quad [\varphi_\Lambda L] = \lambda\varphi, \quad [\alpha_\Lambda Q] = S\alpha, \quad [\varphi_\Lambda Q] = -\chi\varphi.$$

And now by skew-symmetry and sesquilinearity we obtain:

$$(5.4.1) \quad [L_\Lambda \alpha] = T\alpha \quad [L_\Lambda \varphi] = (\lambda + T)\varphi$$

$$(5.4.2) \quad [L_\Lambda T\alpha] = (T + \lambda)T\alpha \quad [L_\Lambda S\alpha] = (S + \chi)T\alpha$$

$$(5.4.3) \quad [Q_\Lambda \alpha] = S\alpha \quad [Q_\Lambda \varphi] = (S + \chi)\varphi$$

$$(5.4.4) \quad [Q_\Lambda S\alpha] = -\chi S\alpha.$$

Formula (5.4.1) says that α and φ are primary fields of conformal weight 0 and 1 respectively. With these formulas and using again the Wick formula (3.2.21.1) we obtain

$$\begin{aligned} [L_\Lambda L] &= [L_\Lambda : (T\alpha)\varphi :] = ((T + \lambda)T\alpha)\varphi : + : T\alpha(\lambda + T)\varphi := \\ &= 2\lambda L + : (T(T\alpha))\varphi : + : T\alpha T\varphi := (T + 2\lambda)L, \end{aligned}$$

since the integral term obviously vanishes. For the other commutation relations we compute:

$$\begin{aligned} [Q_\Lambda Q] &= -[Q_\Lambda : (S\alpha)\varphi :] = \chi : (S\alpha)\varphi : + : S\alpha(\chi + S)\varphi : + \int_0^\Lambda [\chi S\alpha_\Gamma \varphi] d\Gamma = \\ &= : S\alpha S\varphi : + \int_0^\Lambda (\eta - \chi)\eta d\Gamma = SQ + \lambda\chi. \end{aligned}$$

Finally for the last commutator we find:

$$\begin{aligned} [L_\Lambda Q] &= -[L_\Lambda : (S\alpha)\varphi :] = - : ((S + \chi)T\alpha)\varphi : - : S\alpha(\lambda + T)\varphi : - \\ &\quad - \int_0^\Lambda [(S + \chi)T\alpha_\Gamma\varphi]d\Gamma = TQ - \chi : (T\alpha)\varphi : + \lambda Q + \int_0^\Lambda (\eta - \chi)\gamma d\Gamma = \\ &\quad = (T + \lambda)Q - \chi L + \frac{\lambda^2}{2}. \end{aligned}$$

According to (5.1.4) this is a conformal $N_W = 1$ SUSY vertex algebra with central charge 3. It is easy to check that this SUSY vertex algebra is indeed strongly conformal.

Example 5.5 ($V(\mathcal{K}_N)$ series). Consider now the Lie subsuperalgebra $K(1|N) \subset W(1|N)$ consisting of those derivations of $\mathbb{C}[X, X^{-1}]$ preserving the form

$$\omega = dx + \sum_{i=1}^N \xi^i d\xi^i,$$

up to multiplication by a function (cf. Example 2.18). Define the following \mathfrak{g} -valued formal distribution:

$$G(Z) = -2\delta(Z, X)\partial_x - (-1)^N \sum_{i=1}^N (D_X^i \delta(Z, X)) D_X^i.$$

It follows from (2.18.1) that its Z -coefficients form a basis of $K(1|N)$. A straightforward computation shows that this formal distribution satisfies the following commutation relation:

$$\begin{aligned} [G(Z), G(W)] &= 2\delta(Z, W)\partial_w G(W) + (4 - N)(\partial_w \delta(Z, W))G(W) + \\ &\quad + (-1)^N \sum_{i=1}^N (D_W^i \delta(Z, W)) D_W^i G(W), \end{aligned}$$

in particular, the pair of \mathfrak{g} -valued formal distributions $(G(Z), G(Z))$ is local. Letting $\mathcal{F} = \left\{ \partial_Z^{j|J} G(Z), j \geq 0, J \subset \{1, \dots, N\} \right\}$, we obtain an $N_K = N$ formal distribution Lie superalgebra $(\mathfrak{g}, \mathcal{F})$.

Let \mathcal{K}_N be the associated $N_K = N$ SUSY Lie conformal algebra. It is generated as a free \mathcal{H} -module by a vector G of parity $N \bmod 2$ satisfying the following Λ -bracket (for the definition of the algebra \mathcal{H} see 4.3)

$$(5.5.1) \quad [G_\Lambda G] = \left(2T + (4 - N)\lambda + \sum_{i=1}^N \chi^i S^i \right) G.$$

When $N \leq 3$, \mathcal{K}_N admits a non-trivial central extension, obtained by adding the term $\frac{\lambda^{3-N}\chi^N}{3}C$ to the RHS of (5.5.1), where C is even, central and satisfies $TC = S^i C = 0$, cf. [FK02].

When $N = 4$, \mathcal{K}_4 admits a central extension, obtained by adding the term λC to the RHS of (5.5.1). It follows from [KvdL89], [FK02] that \mathcal{K}_N does not admit central extensions when $N > 4$.

When $N \leq 4$, we let $V(\mathcal{K}_N)$ be the universal enveloping $N_K = N$ SUSY vertex algebra of the central extension of \mathcal{K}_N and define $V(\mathcal{K}_N)^c$ to be its quotient by the ideal $(C - c|0)_{(-1|N)}V(\mathcal{K}_N)$, where $c \in \mathbb{C}$ is called the *central charge*. When

$N \geq 5$, we let $V(\mathcal{K}_N)$ be the universal enveloping $N_K = N$ SUSY vertex algebra of \mathcal{K}_N .

In the case $N = 1$, if we expand the corresponding superfield as

$$G(z, \theta) = G(z) + 2\theta L(z),$$

we find that the fields $G(z)$ and $L(z)$ generate a Neveu Schwarz vertex algebra of central charge c as in Example 2.9.

When $N = 2$ expanding the corresponding superfield as (cf. 2.15.2)

$$G(z, \theta^1, \theta^2) = \sqrt{-1}J(z) + \theta^1 G^{(2)}(z) - \theta^2 G^{(1)}(z) + 2\theta^1 \theta^2 L(z)$$

We find that the corresponding fields J, L, G^\pm satisfy the commutation relations of the $N = 2$ vertex algebra as in Example 2.12.

When $N = 4$ the corresponding $N_K = 4$ SUSY vertex algebra is not simple. Indeed the SUSY Lie conformal superalgebra $\mathcal{K}'_4 \subset \mathcal{K}_4$ generated by $S^i G$, $i = 1, \dots, 4$ is an ideal. The central extension of \mathcal{K}_4 described above restricts to a central extension of \mathcal{K}'_4 whose cocycle is given by:

$$(5.5.2) \quad \alpha_1(S^i G, S^j G) = -\chi^i \chi^j C_1.$$

This SUSY Lie conformal algebra admits another central extension given by (cf. [FK02]):

$$(5.5.3) \quad \alpha_2(S^i G, S^j G) = \chi^1 \chi^2 \chi^3 \chi^4 C_2.$$

We let $V(\mathcal{K}'_4)^{c_1, c_2}$ be the corresponding $N_K = 4$ SUSY vertex algebra with $C_i = c_i$.

Note that $\text{Lie}(\mathcal{K}_N) = K(1|N)$ by definition, while $\text{Lie}(\mathcal{K}_N)_\leq = K(1|N)_\leq$ is the Lie superalgebra of regular vector fields preserving ω up to multiplication by a function. We will denote by $K(1|N)_< = \text{Lie}(\mathcal{K}_N)_< \subset \text{Lie}(\mathcal{K}_N)_\leq$ the Lie superalgebra of regular vector fields, preserving ω up to multiplication by a function, and vanishing at the origin.

A field G satisfying the commutation relations (5.5.1) with a central extension, will be called a *super Virasoro* field.

Definition 5.6. Let $N \leq 4$, an $N_K = N$ SUSY vertex algebra V is called *conformal* if there exists a vector $\tau \in V$ (called the conformal vector) such that the corresponding field $G(Z) = Y(\tau, Z)$ satisfies (5.5.1) with a central extension, and moreover

- $\tau_{(0|0)} = 2T$, $\tau_{(0|e_i)} = \sigma(N \setminus e_i, e_i) S^i$,
- the operator $\tau_{(1|0)}$ acts diagonally with eigenvalues bounded below and finite dimensional eigenspaces.

If moreover, the representation of $\text{Lie}(\mathcal{K}_N)_<$ can be exponentiated to the group of automorphisms of the disk $D^{1|N}$ preserving the SUSY structure

$$\omega = dx + \sum_{i=1}^N \xi^i d\xi^i,$$

we will say that V is *strongly conformal*. This amounts to the extra condition

- the operator $\tau_{(1|0)}$ has integer eigenvalues and, if $N = 2$, the operator $\sqrt{-1}\tau_{(0|N)}$ has integer eigenvalues.

If a vector a in a conformal $N_K = N$ SUSY vertex algebra V is an eigenvector of $\tau_{(1|0)}$ with eigenvalue 2Δ , we say that a has *conformal weight* Δ . This happens iff a satisfies

$$[G_\Lambda a] = \left(2T + 2\Delta\lambda + \sum_{i=1}^N \chi^i S^i \right) a + O(\Lambda^2),$$

where $O(\Lambda^2)$ denotes a polynomial in Λ with vanishing constant and linear terms. If, moreover, a satisfies

$$[G_\Lambda a] = \left(2T + 2\Delta\lambda + \sum_{i=1}^N \chi^i S^i \right) a,$$

we say that a is *primary*. For example, formula (5.5.1) with a central extension, says that G has conformal weight $2 - N/2$, and it is primary if the central extension is trivial. As in the $N_W = N$ case, the conformal weight is an important *book-keeping device*

$$\Delta(Ta) = \Delta(a) + 1, \quad \Delta(S^i a) = \Delta(a) + \frac{1}{2}, \quad \Delta(a_{(j|J)} b) = \Delta(a) + \Delta(b) - j - 1 + \frac{N - \#J}{2}.$$

(Note that $\Delta(ab) = \Delta(a) + \Delta(b)$.) Furthermore, letting $\Delta(\lambda) = 1$ and $\Delta(\chi^i) = 1/2$, all terms in $[a_\Lambda b]$ have conformal weight $\Delta(a) + \Delta(b) - 1 + N/2$.

Remark 5.7. The $N_K = N$ SUSY vertex algebra $V(\mathcal{K}_N)^c$ defined in Example 5.5 is strongly conformal when $N < 4$, and $V(\mathcal{K}'_4)^{c_1, c_2}$ is strongly conformal as well.

Example 5.8. (Free fields) The well-known *boson-fermion* system is an $N_K = N$ vertex algebra generated by one superfield. Let Ψ be a vector of parity $(-1)^N$, C an even vector, and define a $N_K = N$ SUSY Lie conformal algebra generated by Ψ and C where C is central, satisfies $TC = S^i C = 0$ and the remaining commutation relations are:

$$[\Psi_\Lambda \Psi] = \Lambda^{1|N} C,$$

when N is even, and

$$[\Psi_\Lambda \Psi] = \Lambda^{0|N} C,$$

when N is odd. Skew symmetry is clear and the Jacobi identity is obvious since all triple brackets vanish. We let \tilde{B}_N be the corresponding universal enveloping $N_K = N$ SUSY vertex algebra, and let B_N be its quotient by the ideal $(C - |0\rangle)_{(-1|N)} \tilde{B}_N$.

To show an application of the above formalism, as well as the subtleties involved in calculations, we will show explicitly that the $N_K = 1$ SUSY vertex algebra B_1 is conformal, the corresponding super Virasoro field being

$$G =: (S\Psi)\Psi : + mT\Psi, \quad m \in \mathbb{C}.$$

Indeed, from sesquilinearity (4.11.1) and skew-symmetry (4.11.2) we find

$$[\Psi_\Lambda S\Psi] = (S + \chi)\chi = \lambda, \quad [S\Psi_\Lambda \Psi] = -\lambda,$$

where we used $[S, \chi] = 2\lambda$ and $\chi^2 = -\lambda$. Using sesquilinearity once more we get:

$$[S\Psi_\Lambda S\Psi] = \chi\lambda, \quad [\Psi_\Lambda T\Psi] = \lambda\chi.$$

Now we can use the $N_K = 1$ version of the non-commutative Wick formula (3.2.21.1) to find:

$$\begin{aligned} [\Psi_\Lambda G] &= (\lambda + \chi S)\Psi + m\lambda\chi, & [S\Psi_\Lambda G] &= \lambda(\chi - S)\Psi - m\lambda^2 \\ [T\Psi_\Lambda G] &= -\lambda(\lambda + \chi S)\Psi - m\lambda^2\chi, \end{aligned}$$

where we note that all the integral terms vanish. Using skew-symmetry again we get:

$$(5.8.1) \quad \begin{aligned} [G_\Lambda \Psi] &= (\lambda + 2T + \chi S)\Psi - m\lambda\chi, & [G_\Lambda S\Psi] &= (\lambda + T)(\chi + 2S)\Psi - m\lambda^2 \\ [G_\Lambda T\Psi] &= (T + \lambda)(\lambda + 2T + \chi S)\Psi - m\lambda^2\chi. \end{aligned}$$

With these formulas we can use again (3.2.21.1) to get:

$$\begin{aligned} [G_\Lambda G] &= ((\lambda + T)(\chi + 2S)\Psi - m\lambda^2) \Psi : + \\ &+ : S\Psi (\lambda + 2T + \chi S)\Psi - m\lambda\chi : + m(T + \lambda)(\lambda + 2T + \chi S)\Psi - m^2\lambda^2\chi \end{aligned}$$

where again the integral term is easily seen to vanish. Note that from quasi-commutativity of the normally ordered product we find $:\Psi\Psi := 0$, from where the expression above reduces to:

$$\begin{aligned} 2\lambda : (S\Psi)\Psi : + \chi : (T\Psi)\Psi : + 2 : (S^3\Psi)\Psi : - m\lambda^2\Psi + \\ + : S\Psi ((\lambda + 2T + \chi S)\Psi - m\lambda\chi) : + m(T + \lambda)(\lambda + 2T + \chi S)\Psi - m^2\lambda^2\chi. \end{aligned}$$

Expanding this expression and after a simple cancellation we find

$$[G_\Lambda G] = (2T + 3\lambda + \chi S)G - m^2\lambda^2\chi.$$

Therefore B_1 is a strongly conformal $N_K = 1$ SUSY vertex algebra with central charge $-3m^2$. Note that, by (5.8.1), Ψ has conformal weight $1/2$ (but it is not primary if $m \neq 0$). Expanding the superfield

$$\Psi(Z) = \varphi(z) + \theta\alpha(z),$$

we find easily that

$$[\varphi_\lambda\varphi] = 1, \quad [\alpha_\lambda\alpha] = \lambda,$$

hence the name *boson-fermion* system.

Example 5.9. (Super Currents) Let \mathfrak{g} be a simple or abelian Lie superalgebra with a non-degenerate invariant supersymmetric bilinear form (\cdot, \cdot) . If N is even, then we define a SUSY Lie conformal algebra (either $N_K = N$ or $N_W = N$) generated by \mathfrak{g} with commutation relations:

$$[a_\Lambda b] = [a, b] + (k + h^\vee)(a, b)\lambda \quad \forall a, b \in \mathfrak{g},$$

where $2h^\vee$ is the eigenvalue of the Casimir operator on \mathfrak{g} .

When N is odd we let $\Pi\mathfrak{g}$ be \mathfrak{g} with reversed parity, and for each element $a \in \mathfrak{g}$ we let \bar{a} be the same element thought in $\Pi\mathfrak{g}$. In this case we define a SUSY Lie conformal algebra generated by $\Pi\mathfrak{g}$ with commutation relations:

$$[\bar{a}_\Lambda \bar{b}] = (-1)^a \left(\overline{[a, b]} + (k + h^\vee)(a, b) \sum_{i=1}^N \chi^i \right).$$

We let $V^k(\mathfrak{g})$ be the associated universal enveloping SUSY vertex algebra⁶, either the $N_K = N$ or the $N_W = N$ vertex algebra, the choice will be clear in each context, as well as the value of N .

⁶Here as before, we are considering a central extension of a SUSY Lie conformal algebra, and then we identify the central element with a multiple of the vacuum vector in the universal enveloping SUSY vertex algebra.

When $N = 1$, the corresponding $N_K = 1$ SUSY vertex algebra is strongly conformal, the corresponding conformal vector is

$$\tau = \frac{1}{k + h^\vee} \left(\sum (-1)^{a^i} : (S\bar{a}^i)\bar{b}^i : + \frac{1}{3(k + h^\vee)} \sum ([a^i, a^j], a^r) : \bar{b}^i : \bar{b}^j \bar{b}^r : \right),$$

where $\{a^i\}$ and $\{b^i\}$ are dual bases for \mathfrak{g} with respect to $(,)$. This is known as the *Kac-Todorov* construction [KT85]. The super Virasoro field $Y(\tau, Z)$ has central charge

$$c = \frac{k \dim \mathfrak{g}}{k + h^\vee} + \frac{\dim \mathfrak{g}}{2},$$

and the fields $\bar{a} \in \bar{\mathfrak{g}}$ have conformal weight $1/2$.

Example 5.10. ($N = 2$ vertex algebra) As a vertex algebra it is generated by 4 fields (cf. Example 2.12). As we have seen in Example 5.5, this is a $N_K = 2$ SUSY vertex algebra generated by one field G . On the other hand, the $N = 2$ vertex algebra admits an embedding of the $N = 1$ vertex algebra. Therefore we can view the $N = 2$ vertex algebra as an $N_K = 1$ SUSY vertex algebra. As such, this algebra is generated by two superfields G and J , where G is a super Virasoro field of central charge c and J is even, primary of conformal weight 1. The remaining Λ -bracket is given by:

$$[J_\Lambda J] = G + \frac{c}{3} \lambda \chi.$$

This is computed using the decomposition Lemma 4.6.

Example 5.11. ($N = 4$ vertex algebra) As a vertex algebra it is generated by 8 fields: a Virasoro field, three currents (for the Lie algebra \mathfrak{sl}_2) and four fermions [Kac96, p. 187]. This vertex algebra admits an embedding of the Neveu Schwarz vertex algebra, therefore we can consider a $N_K = 1$ SUSY vertex algebra structure on it. As a $N_K = 1$ vertex algebra, it is of rank $3|1$, generated by an $N = 1$ conformal vector G with central charge c and three even vectors J^i , $i = 1, 2, 3$. Each pair (G, J^i) generates an $N = 2$ vertex algebra, viewed as an $N_K = 1$ SUSY vertex algebra, as in the previous example. The remaining commutation relations are:

$$[J^i_\Lambda J^j] = \sqrt{-1} \varepsilon^{ijk} (S + 2\chi) J^k, \quad i \neq j,$$

where ε is the totally antisymmetric tensor.

This vertex algebra is the universal enveloping vertex algebra of the central extension of the superconformal Lie algebra $S(1|2; 0)$ (cf. [FK02]).

Example 5.12. ($bc - \beta\gamma$ system) This is a $N_K = 1$ SUSY vertex algebra generated by n even fields B^i and n odd fields Ψ^i . The only non-vanishing Λ -brackets (up to skew-symmetry) are:

$$[B^i_\Lambda \Psi^j] = \delta_{ij}.$$

This SUSY vertex algebra is strongly conformal with super Virasoro field

$$G = \sum_{i=1}^n (: (SB^i)(S\Psi^i) : + : (TB^i)\Psi^i :),$$

and central charge $3n$. The fields B^i (resp. Ψ^i) are primary of conformal weight 0 (resp. $1/2$).

Let σ_{ij}^s , $s = 1, 2, 3$, be three $n \times n$ matrices satisfying

$$\sigma^i \sigma^j = \sqrt{-1} \varepsilon^{ijk} \sigma^k, \quad (\sigma^s)^2 = \text{Id}.$$

The fields

$$J^i = \sum_{j,k=1}^n \sigma_{jk}^i : SB^j \Psi^k :, \quad i = 1, 2, 3,$$

together with G generate an $N = 4$ vertex algebra as in the previous example⁷ (cf. [BZHS06]).

Example 5.13. Here we explain briefly the construction of the chiral de Rham complex of a smooth manifold introduced in [MSV99], using the formalism of $N_K = 1$ SUSY vertex algebras [BZHS06]. Let U be a differentiable manifold. Let \mathcal{T} be the tangent bundle of U and \mathcal{T}^* be its cotangent bundle. We let $T = \Gamma(U, \mathcal{T})$ be the space of vector fields on U and $A = \Gamma(U, \mathcal{T}^*)$ be the space of differentiable 1-forms on U . We let $\mathcal{C} = \mathcal{C}^\infty(U)$ be the space of differentiable functions on U . Denote by

$$\langle, \rangle : A \otimes T \rightarrow \mathcal{C}$$

the natural pairing, and, as before, by Π the functor of change of parity.

Consider now an $N_K = 1$ SUSY Lie conformal algebra \mathcal{R} generated by the vector superspace

$$\mathcal{C} \oplus \Pi T \oplus A \oplus \Pi A.$$

That is, we consider differentiable functions (to be denoted f, g, \dots) as even elements, vector fields X, Y, \dots as odd elements, and finally we have two copies of the space of differential forms. For differential forms, which we consider to be even elements, $\alpha, \beta, \dots \in A$, we will denote the corresponding elements of ΠA by $\bar{\alpha}, \bar{\beta}, \dots$. The nonvanishing Λ -brackets in \mathcal{R} are given by (up to skew-symmetry):

$$\begin{aligned} [X_\Lambda f] &= X(f), & [X_\Lambda Y] &= [X, Y]_{\text{Lie}}, \\ [X_\Lambda \alpha] &= \text{Lie}_X \alpha + \lambda \langle \alpha, X \rangle, & [X_\Lambda \bar{\alpha}] &= \overline{\text{Lie}_X \alpha} + \chi \langle \alpha, X \rangle, \end{aligned}$$

where $[\cdot, \cdot]_{\text{Lie}}$ is the Lie bracket of vector fields and Lie_X is the action of X on the space of differential forms by the Lie derivative. The fact that (5.13) satisfies the Jacobi identity is a straightforward computation (cf. 1.8).

We let $V(U)$ be the corresponding universal enveloping $N_K = 1$ SUSY vertex algebra of \mathcal{R} . This vertex algebra is too big. We impose some relations in $V(U)$ as follows. Let 1_U denote the constant function 1 in U . Let $d : \mathcal{C} \rightarrow A$ be the de Rham differential. Define $I(U) \subset V'(U)$ to be the ideal generated by elements of the form:

$$\begin{aligned} : fg : &= -(fg), & : fX : &= -(fX), & : f\alpha : &= -(f\alpha), & : f\bar{\alpha} : &= -(\overline{f\alpha}), \\ 1_U &= |0\rangle, & Tf &= df, & Sf &= \overline{df}. \end{aligned}$$

Define the $N_K = 1$ SUSY vertex algebra

$$\Omega^{\text{ch}}(U) := V(U)/I(U).$$

The following theorem is a reformulation of the corresponding result in [LL05]:

Theorem 5.14.

- (1) *Let $M \subset \mathbb{R}^n$ be an open submanifold. The assignment $U \mapsto \Omega^{\text{ch}}(U)$ defines a sheaf of SUSY vertex algebras Ω_M^{ch} on M .*

⁷Note however that these fields J^i differ from those used in [BZHS06] by a factor of $\sqrt{-1}$.

- (2) For any diffeomorphism of open sets $M' \xrightarrow{\varphi} M$ we obtain a canonical isomorphism of SUSY vertex algebras $\Omega^{\text{ch}}(M) \xrightarrow{\Omega^{\text{ch}}(\varphi)} \Omega^{\text{ch}}(M')$. Moreover, given diffeomorphisms $M'' \xrightarrow{\varphi'} M' \xrightarrow{\varphi} M$, we have $\Omega^{\text{ch}}(\varphi \circ \varphi') = \Omega^{\text{ch}}(\varphi') \circ \Omega^{\text{ch}}(\varphi)$.

This theorem allows one to construct a sheaf of SUSY vertex algebras in the Grothendieck topology on \mathbb{R}^n (generated by open embeddings). This in turn allows one to attach to any smooth manifold M , a sheaf of SUSY vertex algebras Ω_M^{ch} , called the *chiral de Rham complex of M* .

Example 5.15. (Free Fields) We can generalize Examples 5.8, 5.12, and 5.4 as follows. Let $A = A_{\bar{0}} \oplus A_{\bar{1}}$ be a vector superspace, and let $(,)$ be a non-degenerate bilinear form in A . Recall that the bilinear form $(,)$ is said to be of parity $p \in \mathbb{Z}/2\mathbb{Z}$ if $(a, b) = 0$ unless $p(a) + p(b) = p$, and it is supersymmetric (resp. skew-supersymmetric) if $(a, b) = (-1)^{ab}(b, a)$ (resp. $(a, b) = -(-1)^{ab}(b, a)$).

Let $\mathcal{H} = \mathbb{C}[T, S^i]$ in the $N_W = N$ case, and let \mathcal{H} be defined as in 4.3 in the $N_K = N$ case. Let

$$\mathcal{R} = \mathcal{H} \otimes A \oplus \mathbb{C}C,$$

where C is an even element such that $TC = S^i C = 0$. Given a non-zero homogeneous polynomial $Q(\Lambda)$ of degree s (in PBW basis) and parity p , define the following Λ -bracket on $A \oplus \mathbb{C}C$:

$$[a_{\Lambda} b] = Q(\Lambda)(a, b)C, \quad a, b \in A, \text{ and } C \text{ central},$$

and extend it to \mathcal{R} by sesquilinearity. Then the Jacobi identity automatically holds since all triple brackets are zero. Skewsymmetry holds if and only if

$$(a, b) = -(-1)^{ab}(-1)^{N+s}(b, a).$$

Thus, (5.15) defines a structure of a SUSY Lie conformal algebra, provided that (5.15) holds together with the following parity condition:

$$p + p((,)) = N \pmod{2}.$$

Thus, \mathcal{R} is a SUSY Lie conformal algebra if and only if $N + s$ is even (resp. odd) and the bilinear form $(,)$ is supersymmetric (resp. skew-supersymmetric) of parity $(N - p) \pmod{2}$. The corresponding *free field* SUSY vertex algebra $F(A, Q)$ is the quotient of the universal enveloping vertex algebra $V(\mathcal{R})$ by the ideal $(C - |0\rangle)_{(-1|N)}V(\mathcal{R})$.

Example 5.16. (Spin₇ vertex algebra) In [SV95], Shatashvili and Vafa constructed a vertex algebra associated to any manifold with Spin₇ holonomy. This vertex algebra is generated by four fields and comes equipped with an $N = 1$ superconformal vector, therefore we can view it as an $N_K = 1$ SUSY vertex algebra. As such, it is generated by a super Virasoro field G of central charge $1/2$, and a (non-primary) even field X of conformal weight 2. The corresponding Λ -brackets, derived from the OPE in [SV95], using Lemma 4.6, are:

$$\begin{aligned} [G_{\Lambda} X] &= (2T + \chi S + 4\lambda) X + \frac{\chi\lambda}{2} G + \frac{2}{3} \lambda^3, \\ [X_{\Lambda} X] &= \left(\frac{5}{2} T S X + \frac{5}{4} T^2 G + 6 : G X : \right) + \\ &\quad + 8 (\chi T + \lambda S + 2\lambda\chi) X + \frac{15}{4} \lambda (T + \lambda) G + \frac{8}{3} \lambda^3 \chi. \end{aligned}$$

Note that this Λ -bracket is quadratic in the generating fields. Thus, this SUSY vertex algebra is not the universal enveloping SUSY vertex algebra of a SUSY Lie conformal algebra. Expanding these superfields as:

$$G(Z) = G(z) + 2\theta T(z), \quad X(Z) = \tilde{X}(z) + \theta \tilde{M}(z),$$

we obtain the generating fields as in [SV95].

Example 5.17. (Oda's vertex algebra) In [Oda89], Oda constructed a vertex algebra, generated by eight fields, as an extension of the $N = 2$ vertex superalgebra, associated to manifolds with SU_3 holonomy. It carries therefore an $N = 1$ superconformal vector, so we can view this algebra as an $N_K = 1$ SUSY vertex algebra. As such, it is generated by two superfields G, J forming an $N = 2$ vertex algebra of central charge 9, as in Example 5.10, and two odd superfields X^\pm , primary of conformal weight $3/2$. The remaining Λ -brackets are as follows:

$$\begin{aligned} [J_\Lambda X^\pm] &= \pm(S + 3\chi)X^\pm, & [X^\pm_\Lambda X^\pm] &= 0, \\ [X^+_\Lambda X^-] &= (: JG : + : JSJ : + TG + TSJ) + \\ &\quad + \chi(: JJ : + TJ) + \lambda(G + SJ) + 2\lambda\chi J + \lambda^2\chi. \end{aligned}$$

Note that this relations are also quadratic in the generators as in the previous example. Expanding the generating superfields as

$$\begin{aligned} J(Z) &= I(z) + \theta \frac{1}{\sqrt{2}}(\bar{G}(z) - G(z)), & G(Z) &= \frac{1}{\sqrt{2}}(G(z) + \bar{G}(z)) + 2\theta T(z), \\ X^+(Z) &= X(z) + \theta\sqrt{2}Y(z), & X^-(Z) &= \bar{X}(z) + \theta\sqrt{2}\bar{Y}(z), \end{aligned}$$

we obtain the generating fields as in [Oda89].

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DEPARTMENT OF MATHEMATICS, MIT, CAMBRIDGE, MA 02139, USA
E-mail address: `heluani@math.mit.edu`

DEPARTMENT OF MATHEMATICS, MIT, CAMBRIDGE, MA 02139, USA
E-mail address: `kac@math.mit.edu`